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Inaccessibility and Subinaccessibility

In two parts
Part I

Second edition
enriched and improved

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The work presents the first part of second edition of the previous edition of 2000 under the same title containing the proof (in ZF) of the nonexistence of inaccessible cardinals, now enriched and improved.

This part contains the apparatus of subinaccessible cardinals and its basic tools - theories of reduced formula spectra and matrices, disseminators and others - which are used in this proof and is set forth now in their more transparent and simplified form. Much attention is devoted to the explicit and substantial development and cultivation of basic ideas, serving as grounds for all main constructions and reasonings.

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Introduction

The notion of inaccessibility in its various forms belongs to the most important concepts of cultural traditions since ancientry and consists in the representation of an *extraordinary* large phenomenon that cannot be reached by means of "lesser power" than this phenomenon itself.
In this meaning it appears to be one of the basic archetypes of the Mankind.

In the beginning of 20 century this concept have received its sufficiently adequate expression in foundations of mathematics in the following way. Cofinality of a cardinal k is the minimal ordinal $cf(k)$ that can be mapped by some function f into k such that $\sup rng(f) = k$; the cardinal k is named regular *iff* $cf(k) = k$, otherwise singular; it is named weakly inaccessible *iff* it is uncountable, regular and for every $\alpha < k$ the next after α cardinal is still $< k$. The notion of strongly inaccessible cardinal is obtained by replacing the latter condition with the stronger one: $\forall \alpha < k \;\; 2^{\alpha} < k$.

First weakly inaccessible cardinals were introduced by Felix Hausdorf [1] in 1908; Paul Mahlo [2–4] had studied stronger limit points, Mahlo cardinals, in 1911-1913. Later on there appeared strongly inaccessible cardinals introduced by Waclaw Sierpinski, Alfred Tarski [5] and Ernst Zermelo [6] in 1930. Since that starting period a lot of inaccessible cardinals have appeared – weakly compact, indescribable, Erdös cardinals, Jónsson, Rowbottom, Ramsey cardinals, measurable, strong, Woodin, superstrong, strongly compact, supercompact, extendible, Vopěnka's Principle, almost huge, huge, superhuge cardinals etc., forming the hierarchy corresponding to the substantial strengthening of an "extent of inaccessibility" – all of them have received the common name of *large cardinals.*

So, the tradition of inaccessibility is firmly established in foundations of mathematics serving as a confirmation to the faith in the concept that the main trend of nowadays set theory development lies in forming systems of large cardinal hypotheses, natural models for these hypotheses and generic extensions of such models. At any rate, there is firm belief, that:

"All of the large cardinal hypotheses ... can be given more or less convincing justifications ... – convincing enough, at any rate, that almost no one expects them to be refused.

There are other, more problematic, large cardinal hypotheses. For these there is as yet no convincing justification. Far from being refused, however, these hypotheses have led to some extremely beautiful mathematics." (Kunen [7]).

In addition to a very beautiful justification of set theory as a whole this faith is based on a lot of well-known results pointing to close interrelations (by the relative consistency) between large cardinal hypotheses, Axiom of Determinacy, regular set properties, infinitary combinatorics, infinitary languages and others. A big amount of results of this kind is expounded in monographs of Frank Drake [8] and Akihiro Kanamori [9]; the latter, remarkably exhaustive and contemporary, contains also the outstanding demonstration of diverse large cardinal ideas development.
"The investigation of large cardinal hypotheses is indeed a mainstream of modern set theory, and they have been found to play a crucial role in the study of definable sets of reals, in particular their Lebesgue measurability. Although formulated at various stages in the development of set theory and with different incentives, the hypotheses were found to form a *linear* hierarchy reaching up to

an inconsistent extension of motivating concepts. All known set-theoretic propositions have been gauged in this hierarchy in terms of consistency strength, and the emerging structure of implications provides a remarkably rich, detailed and coherent picture of the strongest propositions of mathematics as embedded in set theory." (Kanamori [9]).

This work constitutes the first part of the second edition of [17], which presents the proof of the theorem:

Main theorem (ZF)

There are no weakly inaccessible cardinals.

This proof is derived as a result of using the subinaccessible cardinal apparatus which the author has developed since 1976. The idea of this proof had arose in 1984 (though initial approaches to it have been undertaken by the author since 1973) and in 1996 it has accepted its present form. Its basic tools – formula spectra and subinaccessible cardinals – were developed in [10]; reduced spectra and matrices in [11–13]; disseminators and δ-matrices in [13]; matrix informativeness was developed by the author in [14]; autoexorsizive matrices in [15]. The proof of main theorem was presented for the first time in [15] and its more transparent and complete variant in [16]; the more systematic presentation of all this field is exposed in [17].

However, one can say that this work [17] exposes the material which is too concise and overloaded by the technical side of the matter to the detriment of explicit development of the basic ideas which should advance their technical development.

These circumstances demand the certain preliminary exposition of the field free of such reproaches.

So, the present work constitutes the first part of the second edition of [17], called to overcome them.

The main theorem proof receives in the work its more transparent and simplified form, though caused by some natural improvements in final definition 8.2 [17] of α-function, and zero characteristic is subjected here to the more refined consideration. This circumstance makes possible simplification of the discussion of α-function and its properties. As the result, the final part of the main theorem proof comes in its more natural purified form. But, on the whole, the technical side of the matter here repeats [17] but in the more systematic way.

The special attention is given here to essential preliminary developments and descriptions of ideas of all main constructions and reasonings.
It should be pointed out, that the main theorem proof itself occupies only § 11 in the second part of this work forthcoming. All previous sections are devoted to the development of the basic and special theories of subinaccessibles which, although used for the creation of ideas and techniques of this proof, but have independent value.

Here, as before, we make a review of the material [10–16] which is necessary for this proof. We shall omit technically simple arguments. For set theory, symbols, concepts and other information necessary for the further reasoning, the remarkable text Jech [18] provides the basic development of the subject and much more, so they are assumed to be known and shall be used frequently without comments.

The plan of this work is the following.
To expose development of the idea of the proof, special notions and terms are needed.
Just with this end in view the first part of the work, Chapter I, is devoted to the basic theory developing, strongly necessary for

introducing the material required for creating the idea.
The very special aim of investigations here makes it possible to confine the theory to its the most reduced part.
Thus, the existence of some cardinal k inaccessible in L is assumed and considerations and reasonings are conducted in the structure L_k or in its generic extensions providing the final contradiction.

So, in § 1 the brief outline of the idea of the proof of main theorem in its first approximation is presented without going in any details. However, the precise and thorough implementation of this idea requires the successive involving of new notions and tools, which are demanded on each stage, since the theory goes on as series of successive approximations caused by some insufficiency of the idea in its previous forms. Therefore it receives its more precise form sequentially involving the necessary techniques coming on.
Some other way of representing the idea, that is right from the start in its *final form*, even without details and rigor, would be opaque and unnatural – or it would be shallow and vague.
Also § 1 contains some classical information.

In § 2 *formula spectra* are introduced (definitions 2.3, 2.4). Briefly saying here, the spectrum of a given formula φ of some level n is the function containing *all information* about its truth properties in *all* generic extensions of the structure L_k of certain kind, by means of L_k-generic ultrafilters on Lévy (ω_0, k)-algebra B. Such spectrum has range consisting of *all essential* Boolean values in B of this formula, that is its *Boolean spectrum*, and domain consisting of corresponding ordinals, named *jump ordinals* of this formula, providing these values, that is its *ordinal spectrum.*

Among spectra the so called *universal spectra* are of most interest (definition 2.6), contracting *all* formula ordinal spectra of a

given level n into one (lemma 2.7).
These notions are considered also in their relativized form; in this case the bounding ordinal $\alpha < k$ causes all truth properties of formula φ in all generic extensions of L_k bounded by α, therefore α is named the *carrier* of these spectra.

After this is done, with formula spectra in hand, we introduce in § 3 the central notion of basic theory – the notion of *subinaccessibility* (definition 3.1). The substantial property of subinaccessible ordinal α of level n lies in the *impossibility* of reaching this ordinal "by means of language" of level n. Since formulas truth properties contains in there spectra, it means that this ordinal α includes all formula spectra of level n with constants $< \alpha$ and so can not be reached by such spectra.
One can see here certain parallel with the notion of *inaccessibility* itself (see the note after definition 3.1).
Some simple properties of these notions make possible to introduce the so called *subinaccessibly universal* formula and its spectra, posessing only subinaccessible ordinals of smaller level (definition 3.9). In what follows these spectra provide the basic theory with its favourable tools.

In § 4 the problem of spectrum "complexity" springs up. It is almost evident that this spectrum characteristic must increase unboundedly while its carrier increases up to k. So the more significant aspect is investigated for *reduced spectra* (definition 4.1), received by reducing their Boolean values to some given cardinal χ.
The main result of this section, lemma 4.6 about spectrum type, shows, that under certain natural and productive conditions the order type of subinaccessibly universal spectrum reduced to χ on a carrier α exceeds every ordinal $< \chi^+$ defined below α (more precisely, defined below its some jump ordinals).

Still, such spectra on different carriers can be hardly compared with each other because their domains, ordinal spectra, consist of ordinals increasing unboundedly, while their carriers are increasing up to k.
Therefore in § 5 we turn to *reduced matrices* (definition 5.1) obtained from reduced inaccessibly universal spectra by isomorphic substituting their domains for corresponding ordinals. The term "matrix" is pertinent here because it is possible to use two-dimensional, three-dimensional matrices, etc. for more fine analysis of the formula truth properties (just on this way the author conducted his initial investigations of the problem during long time). The author have proved in 1977 the main result of this section, lemma 5.11, which makes possible to see, that such matrix can contain *all information* about *every* part of constructive universe and, moreover, about its *every* generic extension, bounded by its carrier (and, again, more precisely, bounded by jump ordinals of corresponding spectrum on this carrier). So, it preserves this information when passing from one carrier to another.

Now, as the required notions and terms are formed, the instruments of the proof of main theorem – *matrix functions* – come into play. Such matrix functions are certain sequences of matrices reduced to the so called *complete cardinal* χ^* (definition 5.4), the supremum of universal ordinal spectrum. First the simplest version of such function (definitions 5.7, 5.14) is exposed at the end of § 5. After that it is sharpened by means of supplying its matrices with the so called *disseminators* (definition 6.1), the special cardinals extending information about lower levels of universe up to their carriers (more precisely, up to jump ordinals of corresponding spectra on these carriers).
Therefore § 6 is devoted to the investigation of this disseminator notion in its most poor version in conformity with the main aim; here we expose some methods of obtaining matrices and matrix functions supplied by such disseminators.

This stage concludes the developing of the basic theory. Here the second part of this work forthcoming, Chapters 2 and 3, should be exposed briefly. With the apparatus in hand, developed in the first part, the special theory of matrix δ-functions starts (definitions 7.1, 7.2), providing certain tools for correction of necessary technical sources.

In § 8 they are developed up to α-functions, representing the instrument of the proof in its final form.

In §§ 9, 10 the properties of α-functions are investigated.

This information provides the main theorem proof exposed in § 11. After that some easy consequences of this theorem and of some well-known results are presented in § 12.

Also one should pay special attention on the comment in the end of this work.

Chapter I

Basic Theory: Subinaccessibility, Formula Spectra and Matrices, Disseminators

1 Preliminaries

We shall prove that the system

$$ZF + \exists k \ \ (k \textit{ is weakly inaccessible cardinal})$$

is inconsistent.
In what follows all the reasoning will be carried out in this system.

As it was noted above, the idea of the main theorem proof is rather complicated and untransparent and therefore requires some special analysis.

So, it is best possible to introduce it into three stages sequentially, approaching to it more precisely on each stage by means of developing the corresponding new notions and techniques, called to overcome certain impeding deficiencies of its previous version. Thus it becomes complicated in essence on each stage and receives its final form in § 8.

The brief outline of the idea is the following.

First the idea springs up at the end of § 5 and rests on the formation of matrix functions that are sequences of matrices, reduced to a fixed cardinal.
One can obtain sufficiently adequate conception of such function

if beforehand acquaint himself with formulations of the following notions in outline: the notions of formula spectra (definitions 2.3, 2.4), universal spectrum (definition 2.6, lemma 2.7), subinaccessibility (definition 3.1), subinaccessibly universal spectrum (definition 3.9), reduced spectra and matrices (definitions 4.1, 5.1) and singular matrices (definition 5.7).
On this foundation the simplest matrix function

$$S_{\chi f} = (S_{\chi\tau})_\tau$$

is introduced (definitions 5.13, 5.14) as the sequence of such matrices of special kind.
This function has range consisting of singular matrices reduced to some fixed cardinal χ and defined as minimal, in the sense of Gödel function Od, on corresponding carriers; this property evidently provides its monotonicity, also in the same sense (lemma 5.17 1)). The domain of this function is extraordinarily long and is cofinal to the inaccessible cardinal k (lemma 5.18)
The role of reducing cardinal χ is played further mainly by the complete cardinal χ^* (definition 5.4).

Now the idea of the main theorem proof comes out in its *initial form*:

The required contradiction can be attained by means of creation some matrix function which should possess inconsistent properties: it should be monotone and at the same time it should be deprived its monotonicity.

However, on this stage the direct proof of this function monotonicity is impeded by the following phenomenon: the properties of universe under consideration change after its bounding by carriers of this function values, that is of reduced matrices $S_{\chi\tau}$ (one can see it from the discussion in the end of § 5 after lemma 5.18).

Just in order to get over this obstacle the process of matrix function transformation starts.

The special cardinals, *disseminators*, are introduced. Though presenting original and valuable phenomenon, this notion is used here only for the transformation of the matrix function in view of our aims.
To this end its values, matrices $S_{\chi\tau}$, are supplied with certain disseminators extending required universum properties from lower levels up to their carriers (definitions 6.1, 5.9, 6.9).
After that, in the second part of this work forthcoming, the second approach to the idea of main theorem proof is undertaken and the matrix function transforms to *δ-function,* also defined on the set which is cofinal to k (definitions 7.1, 7.2, lemma 7.6).

But as the result now, vice versa, this new function loses its monotonicity property (one can see it from the discussion of this new situation in the end of § 7 after lemma 7.7).
The way out of this new stage of things lies in the third approach to the idea of the proof, that is in transformation of this last function into its more complicated recursive form – the so called *α-function* (definitions 8.1-8.3) which is also defined on the set cofinal to k (lemma 8.9). This recursive definition is formed in such a way that cases of monotonicity are demanded in the *first* turn (so they are provided by *"unit characteristic"*), while cases of monotonicity violation are allowed in the last turn only for want of anything better (and they are of *"zero characteristic"*).
Thus, the priority belongs to matrix function values of unit characteristic and just cause of that the situations of monotonicity breaking are avoided.

As the result, α-function, at last, delivers the required contradiction: it cannot be monotone (theorem 1) and at the same time it possesses monotonicity (theorem 2).

Such is the idea of the main theorem proof in its outline form; more detailed description draws into consideration too much technical details. Therefore instead of that one should follow more suitable way of assimilating notions mentioned above (up to the end of § 6) without going into details and getting along, may be, with the help of some images. Using another approach (the representation directly in its final form right from the start) one should receive the exposition even more complicated than the foregoing one, or noninformative and hazy.

After that let us turn to realization of the programme sketched above.

Weakly inaccessible cardinals become strongly inaccessible in Gödel constructive class L; let us remind that it is the class of values of Gödel constructive function F defined on the class of all ordinals. Every set $a \in L$ receives its ordinal number

$$Od(a) = \min\{\alpha : F(\alpha) = a\}.$$

If α is an ordinal then L_α denotes the initial segment

$$\{a \in L : Od(a) < \alpha\}$$

of this class. The starting structure in the further reasoning is the countable initial segment

$$\mathfrak{M} = (L_{\chi^0},\ \in,\ =)$$

of the class L serving as the standard model of the theory

$$ZF + V = L + \exists k\ \ (k \textit{ is weakly inaccessible cardinal})$$

Actually, only the finite part of this system will be used here because we shall consider only formulas of limited length, as it will be clear from what follows. Moreover, the countability of this structure is required only for some technical convenience (see below) and it is possible to get along without it.

Further k is the *smallest inaccessible cardinal in* $\mathfrak{M}$. We shall investigate it "from inside", considering the hierarchy of subinaccessible cardinals; the latter are "inaccessible" by means of formulas of certain elementary language. To receive this hierarchy rich enough it is natural to use some rich truth algebra B. It is well-known (Kripke [19]) that every Boolean algebra is embedded in an appropriate (ω_0, μ)-algebra of displacement and therefore it is natural to use as B the sum of the set of such algebras of power k , that is Lévy (ω_0, k)-algebra B.

Namely, let us apply the set $P \in \mathfrak{M}$ of forcing conditions that are functions $p \subset k \times k$ such that for every limit $\alpha < k$ and $n \in \omega_0$

$$\alpha + n \in dom(p) \longrightarrow p(\alpha + n) < \alpha;$$

also let $p(n) \leq n$ for $\alpha = 0$. The relation $\leq$ of partial order is introduced on P:

$$p_1 \leq p_2 \longleftrightarrow p_2 \subseteq p_1.$$

After that P is embedded densely in the Boolean algebra $B \in \mathfrak{M}$, consisting of regular sections $\subseteq P$, which is complete in $\mathfrak{M}$. The relation of partial order $\leq$ is defined on B:

$$A_1 \leq A_2 \longleftrightarrow A_1 \subseteq A_2,$$

and also Boolean operations $\cdot, +, \prod, \sum$ (see [18]) are defined on B. Every condition $p \in P$ is identified with the section

$$[p] = \{p_1 \in P : p_1 \leq p\}$$

and that is why P is isomorphically embedded in B. Hereafter we recall the well-known results of Cohen [20], Lévy [21] (see also Jech [18]).

Lemma 1.1

Algebra B satisfies the k-chain condition, that is every set $X \subseteq B$, $X \in \mathfrak{M}$ consisting of pairwise disjoint Boolean values *has the power* $< k$ *in* $\mathfrak{M}$ *:*

$$(\forall A_1, A_2 \in X \quad A_1 \cdot A_2 = 0) \longrightarrow |X| < k.$$

According to this lemma it is possible to consider instead of values $A \in B$ only sets

$$P_A = \{p \in A : dom(p) \subseteq \chi\}$$

where

$$\chi = \min\{\chi' : \forall p \in A \quad p\,|\chi' \leq A\}$$

(here $p\,|\chi'$ is the restriction of p to χ'). Since $A = \sum P_A$, we shall always identify A and P_A, that is we shall always consider P_A instead of A itself.
Just due to this convention all Boolean values $A \in B$ are sets in L_k, not classes, and this phenomenon will make possible all further reasoning as a whole.

We shall investigate the hierarchy of subinaccessible cardinals with the help of Boolean values in B of some propositions about their properties, that is working inside the Boolean-valued universe $\mathfrak{M}^B$. The countability of the structure $\mathfrak{M}$ is needed here only to shorten the reasoning when using its generic extensions by means of $\mathfrak{M}$-ultrafilters on B. It is possible to get along without it developing the corresponding reasoning in the Boolean-valued universe L^B (see, for example, [18]).

It will be more suitable to produce generic extensions of $\mathfrak{M}$ not by means of ultrafilters but by means of functions. Namely, as $\mathfrak{M}$-generic or Lévy function on k we shall name every function $l \in {}^k k$ such that every set $X \in \mathfrak{M}, X \subseteq B$, which is dense in P, contains some $p \subset l$. All functions of this kind will be denoted by the common symbol l. Obviously, $\mathfrak{M}$-generic ultrafilters G on

B and these functions mutually define each other:

$$l = \cup(P \cap G) \quad , \quad G = \{A \in B : \exists p \in P(p \leq A \wedge p \subset l)\}.$$

In this case the interpretation i_G of the universe $\mathfrak{M}^B$ is denoted by i_l. As usual, if $\underline{a} \in \mathfrak{M}^B$, $a \in \mathfrak{M}[l]$ and $i_l(\underline{a}) = a$, then $\underline{a}$ is named the label or the name of a. By $\|\varphi\|$ as usual is denoted the Boolean value of the proposition φ with constants from $\mathfrak{M}^B$ in algebra B.

For some convenience we introduce the relation $\overset{*}{\in}$: for every $l \in {}^k k$, $A \in B$

$$l \overset{*}{\in} A \longleftrightarrow \exists p \in P(p \subset l \wedge p \leq A).$$

Lemma 1.2

Let l be an $\mathfrak{M}$-generic function on k and $\varphi(a_1..., a_n)$ be a proposition containing constants $a_1..., a_n \in \mathfrak{M}[l]$ with names $\underline{a}_1..., \underline{a}_n$, then

$$\mathfrak{M}[l] \vDash \varphi(a_1..., a_n) \longleftrightarrow l \ \overset{*}{\in} \|\varphi(\underline{a}_1..., \underline{a}_n)\| \, .$$

Lemma 1.3

Let l be an $\mathfrak{M}$-generic function on k, then:

1) $$\mathfrak{M}[l] \vDash ZF + V = L[l] + GCH + k = \omega_1;$$

2) *for every $\chi < k$ let $\chi_1 = \chi$ iff χ is regular and $\chi_1 = (\chi^+)^{\mathfrak{M}}$ iff χ is singular cardinal in $\mathfrak{M}$, then*

$$\mathfrak{M}[l|\chi] \models \forall \alpha < \chi_1 \ \left(|\alpha| \leq \omega_0 \wedge \forall \alpha \geq \chi_1 |\alpha| = |\alpha|^{\mathfrak{M}}\right).$$

Lemma 1.4

Suppose that

$$t \in \mathfrak{M}^B,\ dom(t) \subseteq \{\breve{a} : a \in \mathfrak{M}\} \ \ and\ |rng(t)| < k.$$

Let B_t *be the subalgebra of* B *generated by* $rng(t)$.
Then for every formula φ

$$\| \varphi(t) \| \in B_t.$$

Such is the preliminary information required to start the developing the theory of subinaccessibles. [1)]

2 Formula Spectra

The main instrument of the further reasoning is the notion of a formula spectrum. In this section the basic spectrum theory is exposed, containing the discussion of the most simple spectrum properties. It is possible to introduce this notion in a more general version (for an arbitrary Boolean algebra B and a partially ordered structure $\mathfrak{M}$); however, here it is enough to use its the most poor variant.

Among names from $\mathfrak{M}^B$ the canonical names are distinguished, that are those which give analogous results under any interpretation. For example, such are names $\breve{a}$ of sets $a \in \mathfrak{M}$ which we shall identify with these sets:

$$dom(\breve{a}) = \{\breve{b} : b \in a\}, \quad rng(\breve{a}) = \{1\}.$$

The canonical name of every $\mathfrak{M}$-generic function l on k is the function

$$\underline{l} = \{((\alpha, \beta), \{(\alpha, \beta)\}) : \{(\alpha, \beta)\} \in P\}.$$

It is easy to see that always $i_l(\breve{a}) = a$, $i_l(\underline{l}) = l$.

Let us introduce the following elementary language $\mathcal{L}$ over the standard structure

$$(L_k\,[l]\,,\ \in,\ =,\ l).$$

Its alphabet consists of usual logic symbols : quantors $\forall, \exists$, connectives $\wedge, \vee, \neg, \longrightarrow, \longleftrightarrow$, brackets (,), individual variables $x, y, z...$ (with indices or without them), all names from Boolean-valued universe L_k^B serving as individual constants, and symbols $\in, =, \underline{l}$.
When interpreting this language in the generic extension $L_k\,[l]$ variables run through $L_k\,[l]$, individual constants $a \in L_k^B$ denote $i_l(a)$ and constants $\in, =, \underline{l}$ denote respectively the standard relations of membership, equality and function l. If this extension

is fixed, then constants and their interpretations will be identified as usual.

The formulas of language $\mathcal{L}$ are defined in a usual recursive way beginning with atomic formulas of the form $t_1 = t_2$, $t_1 \in t_2$, where t_1, t_2 are any terms that are also recursively formed of variables, constants $\in L_k^B$ and $\underline{l}$ by sequential superposition. However, following the tradition, some conventional notation, relations and terms will be used in writings of formulas if it will not cause difficulties. For example, the train of m variables or constants $x_1, ..., x_m$ is denoted by $(x_1, ..., x_m)$ or, in short, by $\overrightarrow{a}$; the order relation $x_1 \in x_2$ on the set of ordinals is denoted by $x_1 < x_2$ and so on.

Further formulas will be considered as formulas of the language $\mathcal{L}$ which we shall denote by small letters from the end of Greek alphabet (unless otherwise specified).

Formula φ which has free variables and individual constants forming a train $\overrightarrow{a}$ will be denoted by $\varphi(\overrightarrow{a})$. If in addition φ contains a symbol to which it is necessary to pay attention it should also be noted specifically; for example, the notation $\varphi(\overrightarrow{a}, \underline{l})$ points out that φ contains an occurrence of $\underline{l}$.

As usual, occurrences of quantors $\exists x, \forall x$ in formula φ are named bounded by a term t *iff* they have the form

$$\exists x \ (x \in t \wedge ...), \forall x \ (x \in t \longrightarrow ...);$$

a formula is named bounded *iff* all its quantors are bounded by some terms; it is named prenex formula *iff* all occurrences of its *unbounded* quantors are disposed to the left from occurrences of other quantors and connectives; this train of its unbounded quantors is named its *quantor prefix.*

Formulas of the language $\mathcal{L}$ will be interpreted in generic extensions $L_k[l]$ and therefore we shall name formulas φ, ψ *equivalent* and write $\varphi \longleftrightarrow \psi$ *iff* they are equivalent in the theory ZFC$^-$, that is ZFC with the Power Set Axiom deleted.

Besides that, when interpreting formulas φ, ψ in L_k we shall

name them *constructively equivalent* and use the same notation $\varphi \longleftrightarrow \psi$ *iff* they are equivalent in $\mathrm{ZFC} + V = L$.
We shall name them *generically equivalent* and write $\varphi \dashv\vdash \psi$, *iff*

$$\|\varphi \longleftrightarrow \psi\| = 1$$

for any values from L_k^B of their free variables.
Further the equivalence in ZFC^- will be considered, if the context will not point to some another case; at any rate, the meaning of the notion of equivalence and of the symbol "$\longleftrightarrow$" will be always obviously specified by the context.

Definition 2.1
1) *The class of all formulas* $\varphi(\overrightarrow{a}, \underline{l})$ *in the prenex form that have quantor prefix consisting of* n *maximal blocks of like quantors and begin with* $\exists$ *and also of all formulas that are equivalent to such formulas is denoted by* $\Sigma_n(\overrightarrow{a})$.
The dual class is denoted by $\Pi_n(\overrightarrow{a})$ *and the class* $\Sigma_n(\overrightarrow{a}) \cap \Pi_n(\overrightarrow{a})$ *– by* $\triangle_n(\vec{a})$.
As a result the elementary Levy hierarchy comes out:

$$\{\Sigma_n(\overrightarrow{a});\ \Pi_n(\overrightarrow{a})\}_{n\in\omega_0}.$$

2) *We denote by* $\Sigma_n^{\dashv\vdash}(\overrightarrow{a})$ *the class of all formulas generically equivalent to formulas from* $\Sigma_n(\overrightarrow{a})$.
The dual class is denoted by $\Pi_n^{\dashv\vdash}(\overrightarrow{a})$ *and the class* $\Sigma_n^{\dashv\vdash}(\overrightarrow{a}) \cap \Pi_n^{\dashv\vdash}(\overrightarrow{a})$ *– by* $\Delta_n^{\dashv\vdash}(\overrightarrow{a})$.

The symbols $Q_n(\overrightarrow{a})$, $Q_n^{\dashv\vdash}(\overrightarrow{a})$ *serve as the common notation respectively of the classes*

$$\Sigma_n(\overrightarrow{a}),\ \Pi_n(\overrightarrow{a}),\quad \Sigma_n^{\dashv\vdash}(\overrightarrow{a}),\ \Pi_n^{\dashv\vdash}(\overrightarrow{a}).$$

The natural index n *is named their level and the level of their formulas.*

Considering fixed extension $L_k[l]$, *sets, that are defined in it by formulas from* $Q_n^{\dashv\vdash}(\overrightarrow{a})$, *will be named* $Q_n^{\dashv\vdash}(\overrightarrow{a})$*-sets.*
As a result the hierarchy

$$\{\Sigma_n^{\dashv\vdash}(\overrightarrow{a});\ \Pi_n^{\dashv\vdash}(\overrightarrow{a})\}_{n\in\omega_0}$$

and the corresponding hierarchy of relations come out.
In all the notation the train $\overrightarrow{a}$ *will be omitted if its value is arbitrary or meant by the context.*

Following the tradition, let us assume that every formula under consideration is considered as transformed to the equivalent prenex form of the *minimal level* — equivalent in ZFC^- or constructibly or generically depending on structures where these formulas are interpreted — of course, if the context does not mean some another situation.
Let us also assume that if in the context formulas interpreted in the structure $(L_k, \in, =)$ are considered, then we use all these terms and notation, but without the index $\dashv\vdash$.

Further the classes Q_n, $Q_n^{\dashv\vdash}$ of some fixed level $n > 3$ are considered (if the context does not imply the opposite). This agreement is taken to have in hand further sufficiently large subinaccessible tools (see lemma 3.5 below for example) and also to use some auxiliary formulas, terms, relations and sets defined in L_k directly as additional constants in formulas notation without raising their level. Obviously, in this way can be considered the sets P, B, relations and operations on them mentioned above, and also the following:

1) $On(x)$ - the bounded formula meaning that x is an ordinal:

$$\forall y\in x(\forall z\in x(y\in z\vee z\in y)\wedge\forall z_1\in y(z_1\in x)).$$

Variables and constants bounded by this formula will be also denoted by small letters from the beginning of Greek alphabet: α, β, γ, ... omitting this formula itself. With the help of this formula it is easy to define all the natural numbers, the ordinals ω_0, ω_0+1.... and so on by the corresponding bounded formulas, so we shall use these ordinals as the additional individual constants of the language.
2) $F(x,y)$ – the Δ_1-formula providing the well-known recursive definition in L_k of Gödel function F [22], that is for every $\alpha \in k$, $a \in L_k$:

$$a = F \mid (\alpha+1) \longleftrightarrow L_k \vDash F(\alpha, a).$$

3) $\lessdot$, $\underline{\lessdot}$ – the relations of well-ordering on L_k:

$$a \lessdot b \longleftrightarrow Od(a) < Od(b) \quad ; \quad a \underline{\lessdot} b \longleftrightarrow a \lessdot b \vee a = b.$$

4) $\lhd$, $\unlhd$ – the corresponding relations on $L_k \times k$:

$$a \lhd \beta \longleftrightarrow Od(a) < \beta \wedge On(\beta) \quad ; \quad a \unlhd \beta \longleftrightarrow a \lhd \beta \vee Od(a) = \beta.$$

5) It is not hard to use similarly the analogous Δ_1 -formula $F(x,y,\underline{l})$ providing the recursive definition in $L_k[l]$ of Gödel constructive function F^l relalively to l and to receive the function of ordinal number Od^l and also the relations $\lessdot^l$, $\lhd^l$:

$$Od^l(a) = \min\{\alpha : F^l(\alpha) = a\} \ ;$$

$$a \lessdot^l b \longleftrightarrow Od^l(a) < Od^l(b) \ ; \qquad a \underline{\lessdot}^l b \longleftrightarrow a \lessdot^l b \vee a = b \ ;$$

$$a \lhd^l \beta \longleftrightarrow Od^l(a) < \beta \wedge On(\beta) \ ; \quad a \unlhd^l \beta \longleftrightarrow a \lhd^l \beta \vee Od^l(a) = \beta \ .$$

It is easy to define all these functions and relations mentioned above by Δ_1-formulas respectively in L_k, $L_k[l]$ absolutely relatively to these structures using formulas $F(x,y)$, $F(x,y,\underline{l})$. We shall denote these formulas by the same symbols as functions and

relations defined by them, but replacing the function l by its name $\underline{l}$. The relation $\lessdot$ was used by Addison [23] over continuum and by Kogalovskiy [24] over arbitrary infinite structures of arbitrary levels.
With the help of ordinary Skolem functions techniques it is easy to prove

Lemma 2.2
Suppose a formula φ *contains in the class* Q_n*, then the formulas*

$$\forall x \in y \ \varphi, \quad \exists x \in y \ \varphi$$

are contained in the same class.
Analogously for the class $Q_n^{\dashv\vdash}$*,* $n \geq 1$*, replacing the bounding formula* $x \in y$ *by the bounding formulas* $x \lessdot^{\underline{l}} y$*,* $x \lhd^{\underline{l}} y$*.*

Let us turn to the notion of spectrum. In order to make it more transparent we shall introduce it only for propositions $\varphi(\overrightarrow{a}, \underline{l})$ having the train of individual constants $\overrightarrow{a} = (a_1, ...a_m)$ consisting of ordinal constants (if the context does not point to another case). It is possible to manage without this convention replacing occurrences of each a_i by occurrences of the term $F^{\underline{l}}(\alpha_i)$ for the corresponding ordinal constant α_i.
Let us also assume that every train $\overrightarrow{a} = (\alpha_1, ..., \alpha_m)$ of ordinals $< k$ is identified with the ordinal which is its image under the canonical order isomorphism of ${}^m k$ onto k. The isomorphism of ${}^2 k$ onto k of this kind will be named the *pair function.*
The next notion plays an important role in what follows:

Definition 2.3
For every formula φ *and ordinal* $\alpha_1 \leq k$ *by* $\varphi^{\lhd \alpha_1}$ *is denoted the formula obtained from* φ *by* $\lhd^{\underline{l}}$ *-bounding all its quantors by the ordinal* α_1*, that is by replacing all occurrences of such*

quantors $\exists x, \forall x$ *by the corresponding occurrences of*

$$\exists x\ (x \triangleleft^{\underline{l}} \alpha_1 \wedge ...), \quad \forall x\ (x \triangleleft^{\underline{l}} \alpha_1 \longrightarrow ...).$$

In addition, if $\alpha_1 < k$, *then we say that* φ *is restricted to* α_1 *or relativized to* α_1; *if, in addition, the proposition* $\varphi^{\triangleleft \alpha_1}$ *holds, then we say that* φ *holds below* α_1 *or that* φ *is preserved under restriction or relativization to* α_1.

The same terminology is carried over to all reasoning and constructions having variables and individual constants $\triangleleft^{\underline{l}}$*- bounded by the ordinal* α_1.
We shall consider $\triangleleft$*-restriction instead of* $\triangleleft^{\underline{l}}$ *-restriction in all these notation and notions iff formulas, constructions and reasoning are interpreted in* L_k.
In all such cases α_1 *is named respectively the* $\triangleleft^{\underline{l}}$ *-bounding or* $\triangleleft$ *-bounding ordinal.*
If $\alpha_1 = k$, *then the upper index* $\triangleleft\ \alpha_1$ *is omitted and such formulas, reasoning and constructions are named unrestricted or unrelativized.*

Definition 2.4
1) *Let* $\varphi(\overrightarrow{a}, \underline{l})$ *be a proposition* $\exists x\ \varphi_1(x, \overrightarrow{a}, \underline{l})$ *and* $\alpha_1 \leq k$. *For every* $\alpha < \alpha_1$ *let us introduce the following Boolean values:*

$$A_{\varphi}^{\triangleleft \alpha_1}(\alpha, \overrightarrow{a}) = \left\| \exists x \trianglelefteq^{\underline{l}} \alpha\ \varphi_1^{\triangleleft \alpha_1}(x, \overrightarrow{a}, \underline{l}) \right\|;$$

$$\Delta_{\varphi}^{\triangleleft \alpha_1}(\alpha, \overrightarrow{a}) = A_{\varphi}^{\triangleleft \alpha_1}(\alpha, \overrightarrow{a}) - \sum_{\alpha' < \alpha} A_{\varphi}^{\triangleleft \alpha_1}(\alpha', \overrightarrow{a}).$$

2) *We name the following function* $\mathbf{S}_{\varphi}^{\triangleleft \alpha_1}(\overrightarrow{a})$ *the spectrum of* φ *on the point* $\overrightarrow{a}$ *below* α_1*:*

$$\mathbf{S}_{\varphi}^{\lhd\alpha_1}(\overrightarrow{a}) = \{(\alpha, \Delta_{\varphi}^{\lhd\alpha_1}(\alpha, \overrightarrow{a})) : \alpha < \alpha_1 \wedge \Delta_{\varphi}^{\lhd\alpha_1}(\alpha, \overrightarrow{a}) > 0\}.$$

Projections

$$dom\left(\mathbf{S}_{\varphi}^{\lhd\alpha_1}(\overrightarrow{a})\right), \quad rng\left(\mathbf{S}_{\varphi}^{\lhd\alpha_1}(\overrightarrow{a})\right)$$

are named respectively the ordinal and the Boolean spectra of φ *on the point* $\overrightarrow{a}$ *below* α_1.

3) *If* $(\alpha, \Delta) \in \mathbf{S}_{\varphi}^{\lhd\alpha_1}(\overrightarrow{a})$, *then* α *is named the jump ordinal of this formula and spectra, while* Δ *is named its Boolean value on the point* $\overrightarrow{a}$ *below* α_1.

4) *The ordinal* α_1 *itself is named the carrier of these spectra.*

If a train $\overrightarrow{a}$ is empty, then we omit it in notations and omit other mentionings about it.

To analyze propositions it is natural to use their spectra. We can develop more fine analysis using their two-dimensional, three-dimensional spectra and so on. [2)]

All spectra introduced possess the following simple properties:

Lemma 2.5

Let φ *be a proposition*

$$\exists x\ \varphi_1(x, \overrightarrow{a}, l), \qquad \varphi_1 \in \Pi_{n-1}^{\dashv\vdash}, \quad \alpha_1 \le k,$$

then:

1) $\sup dom\left(\mathbf{S}_{\varphi}^{\lhd\alpha_1}(\overrightarrow{a})\right) < k$;

2) $\mathbf{S}_{\varphi}^{\lhd\alpha_1}(\overrightarrow{a})$, $dom\left(\mathbf{S}_{\varphi}^{\lhd\alpha_1}(\overrightarrow{a})\right)$ *are* Δ_n*-definable, while* $rng\left(\mathbf{S}_{\varphi}^{\lhd\alpha_1}(\overrightarrow{a})\right)$ *is* Σ_n*-definable in* L_k *for* $\alpha_1 = k$. *For* $\alpha_1 < k$ *all these spectra are* Δ_1*-definable;*

3) $\alpha \in dom\left(\mathbf{S}_{\varphi}^{\lhd\alpha_1}(\overrightarrow{a})\right)$ *iff there exists an* $\mathfrak{M}$*-generic function*

$$l \overset{*}{\in} \Delta_{\varphi}^{\lhd\alpha_1}(\alpha, \overrightarrow{a}).$$

Here the important statement 1) comes directly from lemma 1.1. This statement will be frequently used in what follows.

The so called *universal spectrum* is distinguished among all other spectra.
It is well known that the class $\Sigma_n(\overrightarrow{a})$ for $n > 0$ contains the formula which is universal for this class (Tarski [25], see Addison [23]); let us denote it by $U_n^{\Sigma}(\mathfrak{n}, \overrightarrow{a}, \underline{l})$. Hence, it is universal for the class $\Sigma_n^{\dashv\vdash}(\overrightarrow{a})$ also. Its universality means that for any $\Sigma_n^{\dashv\vdash}(\overrightarrow{a})$-formula $\varphi(\overrightarrow{a}, \underline{l})$ there is a natural $\mathfrak{n}$ such that

$$\varphi(\overrightarrow{a}, \underline{l}) \dashv\vdash U_n^{\Sigma}(\mathfrak{n}, \overrightarrow{a}, \underline{l});$$

this $\mathfrak{n}$ is named the Gödel number of φ. The dual formula universal for $\Pi_n^{\dashv\vdash}(\overrightarrow{a})$ is denoted by $U_n^{\Pi}(\mathfrak{n}, \overrightarrow{a}, \underline{l})$. For some convenience we shall use U_n^{Σ} in the form $\exists x\, U_{n-1}^{\Pi}(\mathfrak{n}, x, \overrightarrow{a}, \underline{l})$. In this notation the upper indices $^{\Sigma}$, $^{\Pi}$ will be omitted in the case when they can be restored from the context or arbitrary.
One should recall that universal formulas come out on the following way. Let us consider a formula $\varphi(\overrightarrow{a}, \underline{l})$ and its equivalent prenex form of the *minimal* level e:

$$Q_1 x_1 Q_2 x_2 ..., Q_i x_i \varphi_1 (x_1, x_2 ..., x_i, \overrightarrow{a}, \underline{l}),$$

where $Q_1 x_1 Q_2 x_2 ..., Q_i x_i$ is its quantor prefix and φ_1 contains only bounded quantors; suppose $Q_i = \exists$, $e > 0$. Removing these last bounded quantors to the left we can receive the equivalent formula of the kind

$$Q_1x_1Q_2x_2...,Q_ix_i$$

$$\forall y_1 \in z_1 \exists v_1 \in w_1 \forall y_2 \in z_2 \exists v_2 \in w_2..., \forall y_j \in z_j \exists v_j \in w_j$$

$$\varphi_2\left(x_1, x_2..., x_i, y_1, z_1, v_1, w_1, y_2, z_2, v_2, w_2..., y_j, z_j, v_j, w_j, \overrightarrow{a}, \underline{l}\right),$$

where each of the variables $z_1, w_1..., z_j, w_j$ is bounded by the others or by the term from the train $x_1..., x_i, \overrightarrow{a}$ and φ_2 contains no quantors. Using Skolem functions techniques one can transform it to the equivalent formula $U_\varphi\left(\overrightarrow{a}, \underline{l}\right)$ of the same level e:

$$Q_1x_1Q_2x_2...,Q_ix_i\exists f_1\exists f_2...,\exists f_j \quad \varphi_3\left(x_1, x_2..., x_i, f_1, f_2..., f_j, \overrightarrow{a}, \underline{l}\right),$$

where $f_1..., f_j$ - variable Skolem functions and φ_3 contains only bounded quantors of the standard kind and disposition.
The case $Q_i = \forall$ one should consider in a dual way.

We shall name this formula $U_\varphi\left(\overrightarrow{a}, \underline{l}\right)$ the *preuniversal form* of the formula $\varphi\left(\overrightarrow{a}, \underline{l}\right)$ under consideration. After that it is easy to receive the equivalent formula with blocks of like quantors contracted into one through application of the Pairing Axiom. It remains to apply to formulas like this one the recursive enumaration of their subformulas following after the quantor prefix.

Let us assume in what follows that formulas we introduce are considered in their *preuniversal form* and use for them the same notation (unless the context does not mean another situation). This agreement preserves the equivalence of a formula $\varphi(a, \underline{l}) \in \Sigma_e^{\dashv\vdash}$ with a Gödel number $\mathfrak{n}$ to the universal formula under $\triangleleft^{\underline{l}}$ -restriction:

$$\varphi^{\triangleleft \alpha_1}\left(\overrightarrow{a}, \underline{l}\right) \longleftrightarrow U_e^{\Sigma \triangleleft \alpha_1}\left(\mathfrak{n}, \overrightarrow{a}, \underline{l}\right)$$

for ordinals α_1 of many kinds (for example, for cardinals). Analogously for $\Pi_e^{\dashv\vdash}$ and for generic equivalence.

Definition 2.6

1) *We name as the spectral universal for the class* $\Sigma_n^{\dashv\vdash}$ *formula of level* n *the formula* $u_n^{\Sigma}(\overrightarrow{a},\underline{l})$ *obtained from the universal formula* $U_n^{\Sigma}(\mathfrak{n},\overrightarrow{a},\underline{l})$ *by replacing all occurrences of the variable* $\mathfrak{n}$ *by occurrences of the term* $\underline{l}(\omega_0)$.
The spectral universal for the class $\Pi_n^{\dashv\vdash}$ *formula* $u_n^{\Pi}(\overrightarrow{a},\underline{l})$ *is introduced in the dual way. Thus, we take*

$$u_n^{\Sigma}(\overrightarrow{a},\underline{l}) = \exists x \; u_{n-1}^{\Pi}(x,\overrightarrow{a},\underline{l}),$$

where $u_{n-1}^{\Pi}(x,\overrightarrow{a},\underline{l})$ *is the spectral universal for the class* $\Pi_{n-1}^{\dashv\vdash}$ *formula.*

2) *The Boolean values*

$$A_{\varphi}^{\triangleleft\alpha_1}(\alpha,\overrightarrow{a}), \Delta_{\varphi}^{\triangleleft\alpha_1}(\alpha,\overrightarrow{a}) \text{ and the spectrum } \mathbf{S}_{\varphi}^{\triangleleft\alpha_1}(\overrightarrow{a})$$

of the formula $\varphi = u_n^{\Sigma}(\overrightarrow{a},\underline{l})$ *and its projections are named the universal Boolean values and spectra of the level* n *on the point* $\overrightarrow{a}$ *below* α_1 *and in their notation the index* u_n^{Σ} *is replaced by* n, *that is they are denoted by*

$$A_n^{\triangleleft\alpha_1}(\alpha,\overrightarrow{a}), \quad \Delta_n^{\triangleleft\alpha_1}(\alpha,\overrightarrow{a}), \quad \mathbf{S}_n^{\triangleleft\alpha_1}(\overrightarrow{a}).$$

3) *If* $(\alpha,\Delta) \in \mathbf{S}_n^{\triangleleft\alpha_1}(\overrightarrow{a})$, *then* α *is named the jump ordinal of these formula and spectra, while* Δ *is named its Boolean value on the point* $\overrightarrow{a}$ *below* α_1.

4) *The ordinal* α_1 *is named the carrier of these spectra.*

Everywhere further $\triangleleft^{\underline{l}}$ - or $\triangleleft$-bounding ordinals α_1 are limit cardinals $< k$ (in L_k) or $\alpha_1 = k$ (if the context does not point to the opposite).

Here the term "universal" is justified by the following

Lemma 2.7
For every proposition $\varphi = \exists x\ \varphi_1(x, \overrightarrow{a}, \underline{l})$, $\varphi_1 \in \Pi^{\dashv\vdash}_{n-1}$*:*

$$dom\left(\mathbf{S}^{\lhd\alpha_1}_{\varphi}(\overrightarrow{a})\right) \subseteq dom\left(\mathbf{S}^{\lhd\alpha_1}_{n}(\overrightarrow{a})\right).$$

Proof. For a function $l \in {}^k k$ let l_0 denote every function $\in {}^k k$ taking values:

$$l_0(\alpha) = \begin{cases} l(\alpha) & , \ \alpha < \omega_0 \vee \alpha > \omega_0 + 1; \\ (l(\omega_0), l(\omega_0+1)) & , \ \alpha = \omega_0 + 1; \end{cases} \tag{2.1}$$

the value $l_0(\omega_0)$ is arbitrary here. Evidently l is $\mathfrak{M}$-generic *iff* l_0 possesses the same property. Let $\varphi_0(\overrightarrow{a}, l)$ denote the formula received by the successive replacing every subformula $\underline{l}(t_1) = t_2$ of the formula φ by the bounded subformula:

$$\begin{gathered} \exists y_1, y_2 < \omega_0 (\underline{l}(\omega_0 + 1) = (y_1, y_2) \wedge \\ \wedge ((t_1 < \omega_0 \vee t_1 > \omega_0 + 1) \longrightarrow \underline{l}(t_1) = t_2) \wedge \\ \wedge (t_1 = \omega_0 \longrightarrow y_1 = t_2) \wedge (t_1 = \omega_0 + 1 \longrightarrow y_2 = t_2)). \end{gathered} \tag{2.2}$$

The subformulas of the form $\underline{l}(t_1) \in t_2$, $t_2 \in \underline{l}(t_1)$ are treated analogously. Let $\mathfrak{n}_0$ be the Gödel number of φ_0. For an arbitrary $\mathfrak{M}$-generic function l let us assume that $l_0(\omega_0) = \mathfrak{n}_0$. Obviously, for every $\alpha < \alpha_1$

$$l \overset{*}{\in} A^{\lhd\alpha_1}_{\varphi}(\alpha, \overrightarrow{a}) \longleftrightarrow l_0 \overset{*}{\in} A^{\lhd\alpha_1}_{\varphi_0}(\alpha, \overrightarrow{a}) \longleftrightarrow l_0 \overset{*}{\in} A^{\lhd\alpha_1}_{n}(\alpha, \overrightarrow{a}) .$$

It remains to apply now lemma 2.5 3). ⊣

From here one can imply that universal spectra accumulate while their ordinal constants are increasing. For $\overrightarrow{a} = (\alpha_1 ..., \alpha_m)$ let $\max \overrightarrow{a} = \max\{\alpha_1 ..., \alpha_m\}$.

Lemma 2.8

For every $\overrightarrow{a}_1, \overrightarrow{a}_2 < \alpha_1$*:*

$$\max \overrightarrow{a}_1 < \max \overrightarrow{a}_2 \longrightarrow dom\left(\mathbf{S}_n^{\lhd\alpha_1}(\overrightarrow{a}_1)\right) \subseteq dom\left(\mathbf{S}_n^{\lhd\alpha_1}(\overrightarrow{a}_2)\right).$$

The proof can be carried out by the so called splitting method (see the proof of lemma 4.6 below for example), but this lemma, although clarifying spectrum properties, is not used further and so its proof is omitted here.

3 Subinaccessible Cardinals

Here is developed the theory of subinaccessibility in its basic aspect.

The further reasoning is conducted in L_k (or in $\mathfrak{M}$ if the context does not mean some another situation).
Let us introduce the central notion of subinaccessibility – the inaccessibility by means of our language. The "meaning" of propositions is contained in their spectra and therefore it is natural to define this inaccessibility by means of the spectra of all propositions of a given level:

Definition 3.1
Let $\alpha_1 \leq k$.
We name an ordinal $\alpha < \alpha_1$ *subinaccessible of a level* n *below* α_1 *iff it fulfills the following formula denoted by* $SIN_n^{<\alpha_1}(\alpha)$*:*

$$\forall \overrightarrow{a} < \alpha \quad dom\left(\mathbf{S}_n^{\lhd \alpha_1}(\overrightarrow{a})\right) \subseteq \alpha.$$

The set

$$\{\alpha < \alpha_1 : SIN_n^{<\alpha_1}(\alpha)\}$$

of all these ordinals is denoted by $SIN_n^{<\alpha_1}$ *and they are named* $SIN_n^{<\alpha_1}$*-ordinals.*
As usual, for $\alpha_1 < k$ *we say that subinaccessibility of* α *is restricted by* α_1 *or relativized to* α_1*. For* $\alpha_1 = k$ *the upper indices* $< \alpha_1$*,* $\lhd \alpha_1$ *are dropped.* $\dashv$

Obviously, the cardinal k is *subinaccessible* itself of any level, if we define this notion for $\alpha = \alpha_1 = k$.
So, the comparison of the notions of inaccessibility and subinaccessibility naturally arises in a following way:
The cardinal k is weakly inaccessible, since it is uncountable and cannot be reached by means of smaller powers in sense that it possess two properties: 1) it is regular and 2) it is closed under

operation of passage to next power: $\forall \alpha < k \quad \alpha^+ < k$.
Turning to *subinaccessibility* of an ordinal $\alpha < k$ of the level n (in its unrelativized form for some brevity), one can see that the property of regularity is dropped now, but α still can not be reached, but by another more powerful means: the second condition is strengthened and now α is closed under more powerful operations of passage to jump ordinals of universal spectrum:

$$\forall \overrightarrow{a} < \alpha \quad \forall \alpha' \in dom\mathbf{S}_n(\overrightarrow{a}) \quad \alpha' < \alpha,$$

that is by means of ordinal spectra *of all* propositions of level n (see lemma 2.7 above).
It implies the closure of α *under all* $\Pi^{\dashv\vdash}_{n-1}$*-functions in all generic extensions* of L_k, not only under operation of power successor in L_k (see lemma 3.5 below).

It is evident that working in L_k one should treat the formula $SIN_n^{<\alpha_1}(\alpha)$ actually as two formulas: one of them without the constant α_1 when $\alpha_1 = k$, and another containing α_1 when $\alpha_1 < k$; the same remark concerns all formulas, constructions and reasoning containing some parameter $\alpha_1 \leq k$.

From definition 3.1 and lemma 2.7 obviously comes

Lemma 3.2 (About restriction)
Let $\alpha < \alpha_1 \leq k$, $\alpha \in SIN_n^{<\alpha_1}$ *and a proposition* $\exists x\, \varphi(x, \overrightarrow{a}, \underline{l})$ *has* $\overrightarrow{a} < \alpha$, $\varphi \in \Pi^{\dashv\vdash}_{n-1}$, *then for every* $\mathfrak{M}$*-generic* l

$$L_k[l] \vDash \left(\exists x \lhd^l \alpha_1 \; \varphi^{\lhd \alpha_1}(x, \overrightarrow{a}, l) \longrightarrow \exists x \lhd^l \alpha \; \varphi^{\lhd \alpha_1}(x, \overrightarrow{a}, l) \right) \;,$$

In this case we say that below α_1 *the ordinal* α *restricts or*

relativizes the proposition $\exists x\ \varphi$.
Considering the same in the inverted form for $\varphi \in \Sigma_{n-1}^{\dashv\vdash}$:

$$L_k\,[l] \vDash \Big(\forall x \triangleleft^l \alpha\ \varphi^{\triangleleft \alpha_1}(x, \overrightarrow{a}, l) \longrightarrow \forall x \triangleleft^l \alpha_1 \varphi^{\triangleleft \alpha_1}(x, \overrightarrow{a}, l) \Big) \ ,$$

we say that below α_1 *the cardinal* α *extends or prolongs the proposition* $\forall x\ \varphi$ *up to* α_1.

Of course, lemma 3.2 presents the stronger statement, the criterion of $SIN_n^{<\alpha_1}$-subinaccessibility. [3)]

Now the following lemmas 3.3 – 3.8 can be easily deduced from definition 3.1 and lemma 2.5 :

Lemma 3.3
The formula $SIN_n^{<\alpha_1}(\alpha)$ *belongs to the class* Π_n *for* $\alpha_1 = k$ *and to the class* Δ_1 *for* $\alpha_1 < k$.

Lemma 3.4
For every $n > 0$:

1) *the set* $SIN_n^{<\alpha_1}$ *is closed in* α_1, *that is for any* $\alpha < \alpha_1$

$$\sup(\alpha \cap SIN_n^{<\alpha_1}) \in SIN_n^{<\alpha_1};$$

2) *the set* SIN_n *is unbounded in* k, *that is* $\sup SIN_n = k$;

3) $$SIN_n^{<\alpha_1}(\alpha) \longleftrightarrow SIN_n^{\triangleleft \alpha_1}(\alpha).$$

Lemma 3.5

Let $\alpha \in SIN_n^{<\alpha_1}$ *and a function* $f \subset \alpha_1 \times \alpha_1$ *be defined in* $L_k[l]$ *by a formula* $\varphi^{\lhd\alpha_1}(\beta,\gamma,l)$ *where* $\varphi \in \Pi_{n-1}^{\dashv\vdash}$*,*
then α *is closed under* f*.*
In particular, for every $n \geq 2$

$$\text{if} \quad \alpha \in SIN_n \quad \text{then} \quad \alpha = \omega_\alpha \quad (\text{in } L_k).$$

The following lemmas 3.6-3.8 represents the important technical tools of subinaccessibility investigations.

Lemma 3.6

For every $m < n$
1)
$$SIN_n^{<\alpha_1} \subset SIN_m^{<\alpha_1};$$

2) moreover, every $\alpha \in SIN_n^{<\alpha_1}$ *is a limit ordinal in* $SIN_m^{<\alpha_1}$ *:*

$$sup\left(\alpha \cap SIN_m^{<\alpha_1}\right) = \alpha$$

Proof. Statement 1) is obvious, because every Σ_m-formula is at the same time Σ_n-formula.
Turning to 2) let us consider the Σ_n-formula

$$\exists\gamma(\beta < \gamma \wedge SIN_m(\gamma))$$

with arbitrary constant $\beta < \alpha$. This formula is true below α_1, because due to 1) the ordinal α itself can be used as γ. After that $SIN_n^{<\alpha_1}$-ordinal α restricts this formula and some $SIN_m^{<\alpha_1}$-ordinal $\gamma > \beta$ appears below α. ⊣

It is obvious that the converse statement is false: subinaccessible cardinals sometimes lose this property on the next level – for example, all the successors in a given class $SIN_m^{<\alpha_1}$.

Lemma 3.7

Let

$$\overrightarrow{a} < \alpha_2 < \alpha_1 \leq k, \quad \alpha_2 \in SIN_n^{<\alpha_1}$$

then for any $Q_n^{\dashv\vdash}$*-proposition* $\varphi(\overrightarrow{a}, \underline{l})$

$$\varphi^{\lhd\alpha_1}(\overrightarrow{a}, \underline{l}) \dashv\vdash \varphi^{\lhd\alpha_2}(\overrightarrow{a}, \underline{l}).$$

Proof. Let us consider the proposition $\varphi = \exists x \quad \varphi_1(x, \overrightarrow{a}, \underline{l})$, $\varphi_1 \in \Pi_{n-1}$, and the ordinal

$$\alpha_0 = min\left\{\alpha : L_k[l] \models \varphi_1^{\lhd\alpha_1}\left(F^l(\alpha), \overrightarrow{a}, l\right)\right\}$$

for an $\mathfrak{M}$-generic l. By lemmas 2.5 3), 2.7 $\alpha_0 \in dom\,(\mathbf{S}_n^{\lhd\alpha_1}(\overrightarrow{a}))$. Since $\alpha_2 \in SIN_n^{<\alpha_1}$ it implies $\alpha_0 < \alpha_2$ and therefore the $\lhd^l$-restriction α_1 in the proposition $\varphi_1^{\lhd\alpha_1}\left(F^l(\alpha_0), \overrightarrow{a}, l\right)$ can be replaced by the $\lhd^l$-restriction α_2. Hence

$$L_k[l] \models (\varphi^{\lhd\alpha_1}(\overrightarrow{a}, l) \longrightarrow \varphi^{\lhd\alpha_2}(\overrightarrow{a}, l)) \ .$$

It remains to convert this argument. ⊣

Lemma 3.8

Let $\alpha_2 < \alpha_1 \leq k$, *then:*

1) *If* $\alpha_2 \in SIN_{n-1}^{<\alpha_1}$, *then the set* $SIN_n^{<\alpha_1} \cap \alpha_2$ *constitutes the initial segment of the set* $SIN_n^{<\alpha_2}$, *that is:*

(i) $SIN_n^{<\alpha_1} \cap \alpha_2 \subseteq SIN_n^{<\alpha_2}$;

(ii) $SIN_n^{<\alpha_2} \cap sup\,(SIN_n^{<\alpha_1} \cap \alpha_2) \subseteq SIN_n^{<\alpha_1}$.

2) *If* $\alpha_2 \in SIN_n^{<\alpha_1}$, *then for every* $m \leq n$

$$SIN_m^{<\alpha_1} \cap \alpha_2 = SIN_m^{<\alpha_2}.$$

Proof. Statements of this lemma one can prove in a similar way and so we shall demonstrate it for 1.(i).
Let $\alpha \in SIN_n^{<\alpha_1} \cap \alpha_2$; it should be proved, that for every $\overrightarrow{a} < \alpha$

$$dom\,(\mathbf{S}_n^{\triangleleft\alpha_2}(\overrightarrow{a})) \subseteq \alpha,$$

so let $\beta \in dom\,(\mathbf{S}_n^{\triangleleft\alpha_2}(\overrightarrow{a}))$. Now we have $\beta < \alpha_2, \overrightarrow{a} < \alpha_2$ and $\alpha_2 \in SIN_{n-1}^{<\alpha_1}$, that is why for every $\beta' \leq \beta$ we can replace the $\triangleleft^{\underline{l}}$ -boundary α_2 in the proposition $u_{n-1}^{\Pi\triangleleft\alpha_2}\left(F^{\underline{l}}(\beta'), \overrightarrow{a}, \underline{l}\right)$ with the $\triangleleft^{\underline{l}}$ -boundary α_1. From here and definition 3.1 it comes $\beta \in dom\,(\mathbf{S}_n^{\triangleleft\alpha_1}(\overrightarrow{a}))$ and then $\alpha \in SIN_n^{<\alpha_1}$ implies $\beta < \alpha$. It means that $dom\,(\mathbf{S}_n^{\triangleleft\alpha_2}(\overrightarrow{a})) \subseteq \alpha$ and hence $\alpha \in SIN_n^{<\alpha_2}$. ⊣

When formulas are equivalently transformed their spectra can change. It is possible to use this phenomenon for the analysis of subinaccessible cardinals. To this end we shall introduce the universal formulas with ordinal spectra containing only subinaccessible cardinals of smaller level. For more clearness of constructions formulas without individual constants will be considered. Let us start with the spectral universal formula for the class $\Sigma_n^{\dashv\vdash}$. The upper indices $^{\Sigma}$, $^{\Pi}$ will be omitted as usual (if it will not cause misunderstanding).
In what follows it is sufficient to consider bounding ordinals only from the class SIN_{n-2}, therefore everywhere further $\triangleleft^{\underline{l}}$- or $\triangleleft$-bounding ordinals α are assumed to be SIN_{n-2}-ordinals or $\alpha = k$ (if the context does not mean another case).
Thus all such ordinals $\alpha \leq k$ are cardinals $\alpha = \omega_\alpha$ due to lemma 3.5.

Definition 3.9
1) *We name as the monotone spectral universal for the class* $\Sigma_n^{\dashv\vdash}$ *formula of the level* n *the* Σ_n*-formula*

$$\widetilde{u}_n(\underline{l}) = \exists x \; \widetilde{u}_{n-1}(x,\underline{l})$$

where $\widetilde{u}_{n-1}(\underline{l}) \in \Pi_{n-1}$ *and*

$$\widetilde{u}_{n-1}(x,\underline{l}) \dashv\vdash \exists x' \lhd^{\underline{l}} x \; u^{\Pi}_{n-1}(x',\underline{l}).$$

2) *We name as the subinaccessibly universal for the class* $\Sigma_n^{\dashv\vdash}$ *formula of the level* n *the* Σ_n*-formula*

$$\widetilde{u}_n^{\sin}(\underline{l}) = \exists x \; \widetilde{u}_{n-1}^{\sin}(x,\underline{l})$$

where $\widetilde{u}_{n-1}^{\sin} \in \Pi_{n-1}$ *and*

$$\widetilde{u}_{n-1}^{\sin}(x,\underline{l}) \dashv\vdash SIN_{n-1}(x) \wedge \widetilde{u}_{n-1}(x,\underline{l}).$$

The monotone and subinaccessibly universal for the class $\Pi_n^{\dashv\vdash}$ *formulas are introduced in the dual way.*

3) *The Boolean values*

$$A_{\varphi}^{\lhd\alpha_1}(\alpha),\; \Delta_{\varphi}^{\lhd\alpha_1}(\alpha), \quad \text{and the spectrum} \quad \mathbf{S}_{\varphi}^{\lhd\alpha_1}$$

of the formula $\varphi = \widetilde{u}_n^{\sin}$ *and its projections are named subinaccessibly universal of the level* n *below* α_1 *and are denoted respectively by*

$$\widetilde{A}_n^{\sin\lhd\alpha_1}(\alpha), \quad \widetilde{\Delta}_n^{\sin\lhd\alpha_1}(\alpha), \quad \widetilde{\mathbf{S}}_n^{\sin\lhd\alpha_1}.$$

4) *If* $(\alpha,\Delta) \in \widetilde{\mathbf{S}}_n^{\sin\lhd\alpha_1}$, *then* α *is named the jump ordinal of these formula and spectra, while* Δ *is named its Boolean value*

below α_1.

5) *The cardinal* α_1 *is named the carrier of these spectra.*

From this definition and lemmas 3.4 2) (for $n-1$ instead of n), 2.5 3), 2.7 and 3.8 the following easy lemmas come:

Lemma 3.10

1)
$$u_n^{\Sigma}(\underline{l}) \dashv\vdash \widetilde{u}_n^{\sin}(\underline{l}) \ .$$

2)
$$dom\left(\widetilde{\mathbf{S}}_n^{\sin\lhd\alpha_1}\right) \subseteq SIN_{n-1}^{<\alpha_1} \cap dom\left(\mathbf{S}_n^{\lhd\alpha_1}\right) \ .$$

3) *Let*

$$\alpha \in dom\left(\mathbf{S}_n^{\lhd\alpha_1}\right) \quad \text{and} \quad \alpha' = min\left\{\alpha'' \in SIN_{n-1}^{<\alpha_1} : \alpha'' > \alpha\right\} \ ,$$

then

$$\alpha' \in dom\left(\widetilde{\mathbf{S}}_n^{\sin\lhd\alpha_1}\right) \ .$$

4) *Let*

$$\alpha \leq \alpha_1 \quad \textit{be limit in} \quad SIN_{n-1}^{<\alpha_1} \ ,$$

then

$$\sup dom\left(\widetilde{\mathbf{S}}_n^{\sin\lhd\alpha_1}|\alpha\right) = \sup dom\left(\mathbf{S}_n^{\lhd\alpha_1}|\alpha\right).$$

Lemma 3.11

Let

$$\alpha_2 \in SIN_{n-2}^{<\alpha_1} \quad \textit{and} \quad \alpha_0 = \sup\left(SIN_{n-1}^{<\alpha_1} \cap \alpha_2\right),$$

then

$$\widetilde{\mathbf{S}}_n^{\sin\lhd\alpha_2}|\alpha_0 = \widetilde{\mathbf{S}}_n^{\sin\lhd\alpha_1}|\alpha_0 \quad .$$

Proof. Let $\alpha < \alpha_0, (\alpha, \Delta) \in \widetilde{\mathbf{S}}_n^{\sin \triangleleft \alpha_2}$ and hence

$$\Delta = \widetilde{\Delta}_n^{\sin \triangleleft \alpha_2}(\alpha) > 0.$$

For every $\alpha' \leq \alpha$ by lemma 3.10 2) and lemma 3.8 (used for $n-1$ instead of n)

$$SIN_{n-1}^{<\alpha_2}(\alpha') \longleftrightarrow SIN_{n-1}^{<\alpha_1}(\alpha')$$

and so the $\triangleleft^{\underline{l}}$ -boundary α_2 in the proposition $\widetilde{u}_{n-1}^{\triangleleft \alpha_2}\left(F^{\underline{l}}(\alpha'), \underline{l}\right)$ can be replaced with the $\triangleleft^{\underline{l}}$ -boundary α_1. It implies

$$\Delta = \widetilde{\Delta}_n^{\sin \triangleleft \alpha_1}(\alpha) \quad \textit{and} \quad (\alpha, \Delta) \in \widetilde{\mathbf{S}}_n^{\sin \triangleleft \alpha_1}.$$

The converse argument completes the proof. ⊣

Our aim is to "compare" universal spectra with each other on *different* carriers α_1 disposed cofinally to k in order to introduce *monotone* matrix functions. To this end it is natural to do it by means of using values of function Od for such spectra.
Also it is natural to try to find some estimates of "informational complexity" of these spectra by means of estimates of their order types. But from lemmas 2.7, 2.8, 3.10 it follows that, for instance, spectra $dom\left(\widetilde{\mathbf{S}}_{n-1}^{\sin \triangleleft \alpha_1}(\alpha)\right)$ accumulate when α, α_1 are increasing and therefore their order types increase up to k; in addition they are closed under Π_{n-2}-functions, etc. Spectra $\widetilde{\mathbf{S}}_n^{\sin \triangleleft \alpha_1}$ have analogous properties.
Therefore the required comparison of such spectra can be hardly carried out in a proper natural way since they *are "too much differ"* from each other for arbitrary great carriers α_1.
So, there is nothing for it but to consider further *spectra reduced to some fixed cardinal* and, next, *reduced matrices.*

4 Reduced Spectra

Here we start to form the main material for building matrix functions – reduced matrices.
With this end in view first we shall consider the necessary preliminary constructions – reduced spectra.

For an ordinal $\chi \leq k$ let P_χ denote the set $\{p \in P : dom(p) \subseteq \chi\}$ and B_χ denote the subalgebra of B generated by P_χ in L_k. For every $A \in B$ let us introduce the set

$$A\lceil\chi = \{p \in P_\chi : \exists q \ (p = q|\chi \wedge q \leq A)\}$$

which is named the value of A reduced to χ. It is known (see [18]) that

$$B_\chi = \left\{\sum X : X \subseteq P_\chi\right\}$$

and therefore every $A \in B_\chi$ coincides with $\sum A\lceil\chi$. Therefore let us identify every $A \in B_\chi$ with its reduced value $A\lceil\chi$; so, here one should point out again, that cause of that every value $A \in B_\chi$ is the set in L_k, not class, and B_χ is considered as the set of such values.

Definition 4.1

Let $\chi \leq k$, $\alpha_1 \leq k$.

1) *For every* $\alpha < \alpha_1$ *let us introduce the Boolean values and the spectrum:*

$$\widetilde{A}_n^{\sin\lhd\alpha_1}(\alpha)\lceil\chi;\ \widetilde{\Delta}_n^{\sin\lhd\alpha_1}(\alpha)\overline{\lceil}\chi = \widetilde{A}_n^{\sin\lhd\alpha_1}(\alpha)\lceil\chi - \sum_{\alpha'<\alpha} \widetilde{A}_n^{\sin\lhd\alpha_1}(\alpha')\lceil\chi;$$

$$\widetilde{\mathbf{S}}_n^{\sin\lhd\alpha_1}\overline{\overline{\lceil}}\chi = \{(\alpha;\ \widetilde{\Delta}_n^{\sin\lhd\alpha_1}(\alpha)\overline{\lceil}\chi) : \alpha < \alpha_1 \wedge \widetilde{\Delta}_n^{\sin\lhd\alpha_1}(\alpha)\overline{\lceil}\chi > 0\}.$$

2) *These values, spectrum and its first and second projections are named subinaccessibly universal reduced to* χ *of the level* n

below α_1.
3) *If* $(\alpha, \Delta) \in \widetilde{\mathbf{S}}_n^{\sin\lhd\alpha_1} \overline{\overline{\lceil}} \chi$, *then* α *is named the jump cardinal of these spectra, while* Δ *is named their Boolean value reduced to* χ *below* α_1.
4) *The cardinal* α_1 *is named the carrier of these spectra.*

In a similar way multi-dimensional reduced spectra can be introduced. [4)]
Further it is always assumed that χ is closed under the pair function; if $\chi = k$, all mentionings about χ will be dropped.
It is not difficult to derive the following two lemmas from definitions 3.9, 4.1 and lemmas 3.10, 3.11:

Lemma 4.2

$$dom\left(\widetilde{\mathbf{S}}_n^{\sin\lhd\alpha_1} \overline{\overline{\lceil}} \chi\right) \subseteq dom\left(\widetilde{\mathbf{S}}_n^{\sin\lhd\alpha_1}\right) \subseteq SIN_{n-1}^{<\alpha_1} \cap dom\left(\mathbf{S}_n^{\lhd\alpha_1}\right).$$

Lemma 4.3
Let

$$\alpha_2 \in SIN_{n-2}^{<\alpha_1} \quad \text{and} \quad \alpha_0 = \sup(SIN_{n-1}^{<\alpha_1} \cap \alpha_2),$$

then

$$(\widetilde{\mathbf{S}}_n^{\sin\lhd\alpha_2} \overline{\overline{\lceil}} \chi)|\alpha_0 = (\widetilde{\mathbf{S}}_n^{\sin\lhd\alpha_1} \overline{\overline{\lceil}} \chi)|\alpha_0 \quad .$$

The following lemma is analogous to lemma 2.5 and comes from definitions:

Lemma 4.4
Let $\alpha < \alpha_1$, $\chi \leq k$, *then:*

1) $\sup dom\left(\widetilde{\mathbf{S}}_n^{\sin\lhd\alpha_1} \overline{\overline{\lceil}} \chi\right) < k.$

2) $\widetilde{\mathbf{S}}_n^{\sin\triangleleft\alpha_1}\overline{\overline{\lceil}}\chi$, *dom* $\left(\widetilde{\mathbf{S}}_n^{\sin\triangleleft\alpha_1}\overline{\overline{\lceil}}\chi\right)$ *are* Δ_n*-definable, while*

rng $\left(\widetilde{\mathbf{S}}_n^{\sin\triangleleft\alpha_1}\overline{\overline{\lceil}}\chi\right)$ *is* Σ_n*-definable in* L_k *for* $\alpha_1 = k$.

For $\alpha_1 < k$ *these spectra are* Δ_1*-definable ;*

3) $\alpha \in$ *dom* $\left(\widetilde{\mathbf{S}}_n^{\sin\triangleleft\alpha_1}\overline{\overline{\lceil}}\chi\right)$ *iff there exists an* $\mathfrak{M}$*-generic function*

$$l \stackrel{*}{\in} \widetilde{\Delta}_n^{\sin\triangleleft\alpha_1}(\alpha)\overline{\lceil}\chi;$$

4)
$$\left\|\widetilde{u}_n^{\sin\triangleleft\alpha_1}(\underline{l})\right\|\lceil\chi = \sum rng\left(\widetilde{\mathbf{S}}_n^{\sin\triangleleft\alpha_1}\overline{\overline{\lceil}}\chi\right).$$

As in lemma 2.5 3) statement 3) here makes it possible to discover jump ordinals α with the help of generic functions l; this techniques is used below in the proof of lemma 4.6. But it is possible to get along without it using instead of l conditions $p \subset l$ with a sufficiently long domain.

Let us turn to the discussion of order spectrum types. If X is a well ordered set, then its order type is denoted by $OT(X)$; if X is a function having well ordered domain, then we assume $OT(X) = OT(dom(X))$.

Rough upper estimate of spectrum types comes from lemma 1.1, $|P_\chi| = |\chi|$ and GCH in L_k :

Lemma 4.5

$$OT(\widetilde{\mathbf{S}}_n^{\sin\triangleleft\alpha_1}\overline{\overline{\lceil}}\chi) < \chi^+ \ .$$

Now let us discuss estimates of such types from below. Here comes out the lemma essential for the proof of main theorem. It shows, that as soon as an ordinal $\delta < \chi^+$ is defined through some jump ordinal of the subinaccessibly universal spectrum reduced to

χ, the order type of this spectrum exceeds δ under certain natural conditions.

We shall use here and further the method of reasoning that may be named *splitting method.* In its simplest version it consists in splitting some value

$$\Delta = \widetilde{\mathbf{S}}_n^{\sin \vartriangleleft \alpha_1} \overline{\overline{\lceil}} \chi(\alpha)$$

(or several such values) under consideration in a sequence of its parts and in assigning them to successive $SIN_{n-1}^{<\alpha_1}$-cardinals respectively after their certain slight transformation. After that these cardinals become *jump cardinals* of the spectrum. For this purpose beforehand the cardinal α must be *fixed* by some l, that is an $\mathfrak{M}$-generic function $l \overset{*}{\in} \Delta$ must be used.

For some convenience the suitable notation of ordinal intervals will be used for $\alpha_1 < \alpha_2$:
$[\alpha_1, \alpha_2[= \alpha_2 - \alpha_1$; $]\alpha_1, \alpha_2[= \alpha_2 - (\alpha_1 + 1)$; $[\alpha_1, \alpha_2] = (\alpha_2 + 1) - \alpha_1$; $]\alpha_1, \alpha_2] = (\alpha_2 + 1) - (\alpha_1 + 1)$ (here α_1, α_2 are sets of smaller ordinals).

Lemma 4.6 (About spectrum type)
Let ordinals $\overline{\delta}$, $\overline{\chi}$, $\overline{\alpha}_0$, $\overline{\alpha}_1$ *be such that:*

(i) $\overline{\delta} < \overline{\chi}^+ < \overline{\alpha}_0 < \overline{\alpha}_1 \leq k$;

(ii) $SIN_{n-2}(\overline{\alpha}_1) \wedge OT(SIN_{n-1}^{<\overline{\alpha}_1}) = \overline{\alpha}_1$;

(iii) $SIN_{n-1}^{<\overline{\alpha}_1}(\overline{\chi}) \wedge \sup dom \left(\widetilde{\mathbf{S}}_n^{\sin \vartriangleleft \overline{\chi}} \right) = \overline{\chi}$;

(iv) $\sum rng \left(\widetilde{\mathbf{S}}_n^{\sin \vartriangleleft \overline{\chi}} \right) \in B_{\overline{\chi}}$;

(v) $\overline{\alpha}_0 \in dom \left(\widetilde{\mathbf{S}}_n^{\sin \vartriangleleft \overline{\alpha}_1} \overline{\overline{\lceil}} \overline{\chi} \right)$;

(vi) $\overline{\delta}$ *is defined in* L_k *through ordinals* $\overline{\alpha}_0$, $\overline{\chi}$ *by a for-*

mula of the class $\Sigma_{n-2} \cup \Pi_{n-2}$.

Then $$\overline{\delta} < OT(\widetilde{\mathbf{S}}_n^{\sin \triangleleft \overline{\alpha}_1} \overline{\overline{\lceil}} \overline{\chi}) \quad .$$

Proof. Let us introduce the following formulas describing the essential aspects of the situation below $\overline{\alpha}_1$.
By condition (vi) there exists the $\Sigma_{n-2} \cup \Pi_{n-2}$-formula $\psi_0(\alpha_0, \chi, \delta)$ which defines $\overline{\delta}$ through $\alpha_0 = \overline{\alpha}_0$, $\chi = \overline{\chi}$, that is $\delta = \overline{\delta}$ is the only ordinal satisfying $\psi_0(\overline{\alpha}_0, \overline{\chi}, \delta)$ in L_k.
Due to (iii) and lemma 3.5 and the minimality of k $\overline{\chi}$ is the singular cardinal and then by lemma 1.3 2) for every $\mathfrak{M}$-generic function $l|\overline{\chi}$ on $\overline{\chi}$ ordinals $\overline{\chi}$, $\overline{\delta}$ are countable in $L_k[l|\overline{\chi}]$.
Let us denote by $(f : \omega_0 \longrightarrow \delta + 1)$ the formula

$$\text{“}f \text{ maps } \omega_0 \text{ onto } \delta + 1\text{”} \quad .$$

In algebra $B_{\overline{\chi}}$ it has the Boolean value

$$\| \exists f \ (f : \omega_0 \longrightarrow \delta + 1) \|_{\overline{\chi}} = 1$$

and therefore there exists some name $\underline{f} \in L_k^{B_{\overline{\chi}}}$ for which

$$\| \underline{f} : \omega_0 \longrightarrow \delta + 1 \|_{\overline{\chi}} = 1$$

in $B_{\overline{\chi}}$ (see [18]). Hence there is the ordinal $\overline{\beta}$ defined through $\alpha_0 = \overline{\alpha}_0$, $\chi = \overline{\chi}$ by the following formula which is denoted by $\psi_1(\alpha_0, \chi, \beta)$:

$$\exists \delta < \alpha_0 (\psi_0(\alpha_0, \chi, \delta) \wedge$$
$$\wedge \beta = \min\{\beta' : F(\beta') \in L^{B_\chi} \wedge \left\| F(\beta') : \omega_0 \longrightarrow \delta + 1 \right\|_\chi = 1\}).$$

From conditions (ii), (v) and lemmas 3.8, 4.2 it follows that $\overline{\alpha}_0 \in SIN_{n-2}$ and then $\overline{\beta} < \overline{\alpha}_0$. Next from (ii), (v) and the same lemma 4.2 it follows that $\overline{\alpha}_0 \in dom\left(\widetilde{\mathbf{S}}_n^{\sin \triangleleft \overline{\alpha}_1}\right)$ and it makes possible to define the jump cardinal

$$\overline{\alpha}_0' = \min([\overline{\chi}, \overline{\alpha}_0] \cap dom\left(\widetilde{\mathbf{S}}_n^{\sin \lhd \overline{\alpha}_1}\right)).$$

Since $\overline{\chi}, \overline{\alpha}_0' \in SIN_{n-1}^{<\overline{\alpha}_1}$ from (iii), (iv) it follows that

$$\overline{\alpha}_0' \in dom\left(\widetilde{\mathbf{S}}_n^{\sin \lhd \overline{\alpha}_1} \overline{\overline{\lceil}} \overline{\chi}\right)$$

and the cardinal $\overline{\chi}$ is definable through $\alpha_0' = \overline{\alpha}_0'$ by the formula

$$\chi = \sup dom\left(\widetilde{\mathbf{S}}_n^{\sin \lhd \alpha_0'}\right)$$

which we denote by $\psi_2(\alpha_0', \chi)$.
Let us assume that formulas ψ_i, $i = \overline{0,2}$, are transformed in the preuniversal form; further they will be used in generic extension of L_k and at that time they will be bounded by the constructive class and denoted by ψ_i^L, $i = \overline{0,2}$.

Now the *splitting method* starts to work. To this end the notion of direct product of functions is needed. Let $l_0, ..., l_m \in {}^k k$; on k there is defined the function $l = l_0 \oplus ... \oplus l_m$ in the following way: for every ordinals $\alpha < k$, $\alpha = \alpha_0 + (m+1)i + j$, where the ordinal α_0 is limit and $i \in \omega_0$, $j \leq m$, it has the value $l(\alpha) = l_j(\alpha_0 + i)$. Let us denote by $(,)_j^m$ the operation reconstructing l_j by l, that is $l_j = (l)_j^m$. It is known (see Solovay [26]) that l is the $\mathfrak{M}$-generic function on k *iff* l_0 is the $\mathfrak{M}$-generic function and l_j is the $\mathfrak{M}[l_0 \oplus ... \oplus l_{j-1}]$-generic function on k, $j = \overline{1,m}$. We shall take here $\mathfrak{M}$-generic l_0, l_1 to hold fixed Boolean values of the spectrum on jump cardinals $\overline{\alpha}_0$, $\overline{\alpha}_0'$, that is take l_0, l_1 such that

$$l_0 \overset{*}{\in} \widetilde{\Delta}_n^{\sin \lhd \overline{\alpha}_1}(\overline{\alpha}_0) \overline{\lceil} \chi, \qquad l_1 \overset{*}{\in} \widetilde{\Delta}_n^{\sin \lhd \overline{\alpha}_1}(\overline{\alpha}_0') \overline{\lceil} \chi;$$

there existence comes from lemma 4.4 3).
Now let us consider the formula $\varphi = \exists \alpha\ \varphi_1(\alpha, \underline{l})$ collecting all information about the situation below $\overline{\alpha}_1$ obtained and specifying

the splitting Boolean values on jump cardinals $\overline{\alpha}_0$, $\overline{\alpha}'_0$, where φ_1 is the following formula:

$$SIN_{n-1}(\alpha) \wedge$$

$$\wedge \exists \alpha_0 < \alpha \exists \alpha'_0 \leq \alpha_0 \exists \chi \leq \alpha'_0 \exists \beta < \alpha_0 \exists i \in \omega_0 \exists \delta < \alpha_0 \exists \delta_i \leq \delta \exists p \in P_\chi$$

$$\big[SIN^{<\alpha}_{n-1}(\alpha_0) \wedge SIN^{<\alpha}_{n-1}(\alpha'_0) \wedge \widetilde{u}^{\lhd\alpha}_{n-1}(\alpha_{0,}(\underline{l})^3_0) \wedge \widetilde{u}^{\lhd\alpha}_{n-1}(\alpha'_0, (\underline{l})^3_1) \wedge$$

$$\wedge \psi^L_0(\alpha_0, \chi, \delta) \wedge \psi^L_1(\alpha_0, \chi, \beta) \wedge \psi^L_2(\alpha'_0, \chi) \wedge$$

$$\wedge (\underline{l})^3_2(\omega_0) = i \wedge p \subseteq (\underline{l})^3_3 \wedge p \leq \|F(\beta)(i) = \delta_i\|_\chi \wedge$$

$$\wedge OT\{\alpha' < \alpha : SIN^{<\alpha}_{n-1}(\alpha')\} = \alpha_0 + \delta_i\big] .$$

This formula belongs to Π_{n-1} since all its variables in square brackets are bounded by the SIN_{n-1}-variable α. For every $\delta \leq \overline{\delta}$ let us denote by α_δ the cardinal $\alpha < \overline{\alpha}_1$ such that

$$SIN^{<\overline{\alpha}_1}_{n-1}(\alpha) \wedge OT\{\alpha' < \alpha : SIN^{<\alpha}_{n-1}(\alpha')\} = \overline{\alpha}_0 + \delta \; ; \qquad (4.1)$$

this cardinal does exist due to *(ii)* and lemma 3.5. Let us show that every α_δ is the jump cardinal of φ below $\overline{\alpha}_1$ by means of *splitting* Boolean values mentioned above. To this end let us assume $\mathfrak{M}$-generic function on k $l = l_0 \oplus l_1 \oplus l_2 \oplus l_3$ such that

$$L_k\,[l_0] \vDash \widetilde{u}^{\lhd\overline{\alpha}_1}_{n-1}(\overline{\alpha}_{0,}l_0) \quad ; \qquad (4.2)$$

$$l_0 \overset{*}{\notin} \left\| \exists \alpha < \overline{\alpha}_0 (SIN^{\lhd\overline{\alpha}_1}_{n-1}(\alpha) \wedge \widetilde{u}^{\lhd\overline{\alpha}_1}_{n-1}(\alpha_{,}\underline{l})) \right\| \lceil \overline{\chi} \quad ; \qquad (4.3)$$

$$L_k\,[l_1] \vDash \widetilde{u}^{\lhd\overline{\alpha}_1}_{n-1}(\overline{\alpha}'_0, l_1) \quad ; \qquad (4.4)$$

$$l_1 \overset{*}{\notin} \left\| \exists \alpha < \overline{\alpha}'_0 (SIN_{n-1}^{\lhd \overline{\alpha}_1}(\alpha) \wedge \widetilde{u}_{n-1}^{\lhd \overline{\alpha}_1}(\alpha, \underline{l})) \right\| \lceil \overline{\chi} \quad ; \tag{4.5}$$

$$l_2(\omega_0) = i_\delta \quad ; \tag{4.6}$$

$$\exists p \in P_{\overline{\chi}} (p \subset l_3 \wedge p \leq \left\| F(\overline{\beta})(i_\delta) = \delta \right\|_{\overline{\chi}}) \tag{4.7}$$

for corresponding $i_\delta \in \omega_0$. Here the existence of l_2, l_3 is obvious. From (4.2), (4.4), (4.6), (4.7) and (4.1) it comes

$$l \overset{*}{\in} \left\| \varphi_1^{\lhd \overline{\alpha}_1}(\alpha_\delta, \underline{l}) \right\| \lceil \overline{\chi} \quad . \tag{4.8}$$

Then from (4.3), (4.5) it is not difficult to deduce that

$$l \overset{*}{\notin} \left\| \exists \alpha < \alpha_\delta \ \varphi_1^{\lhd \overline{\alpha}_1}(\alpha, \underline{l}) \right\| \lceil \overline{\chi} \quad .$$

Along with (4.8) it implies $\alpha_\delta \in dom \left(\mathbf{S}_\varphi^{\lhd \overline{\alpha}_1} \overline{\overline{\lceil}} \overline{\chi} \right)$. Then with the help of condition (ii) it is not hard to see that for every $\delta \leq \overline{\delta}$ $dom \left(\widetilde{\mathbf{S}}_n^{\sin \lhd \overline{\alpha}_1} \overline{\overline{\lceil}} \overline{\chi} \right)$ contains the cardinal succeeding α_δ in $SIN_{n-1}^{<\overline{\alpha}_1}$. ⊣

Complicating this reasoning insignificantly one can prove this lemma in the case when δ is defined through $\overline{\chi}$ and several jump cardinals

$$\overline{\alpha}_1, \ldots, \overline{\alpha}_m \in dom \left(\widetilde{\mathbf{S}}^{\sin \lhd \overline{\alpha}_1} \overline{\overline{\lceil}} \overline{\chi} \right).$$

Lemma 4.6 admits various strengthenings and versions but they are indifferent for what follows and therefore are omitted. [5)]

In this reasoning the formulas interpreted in L_k were equipped with the upper index L when turning to extensions of L_k. Further we shall omit this index if the corresponding passage is meant by the context.

Lemma 4.6 about spectrum type here is the substantially important property of reduced spectra along with their informative properties (see lemma 5.1 below).
But still there is the following essential inconvenience: such spectra, taken on their different carriers, can be hardly compared with each other in view to their basic properties, because their domains can contain an arbitrary great cardinals, when these carriers are increasing up to k. In order to avoid this obstacle we shall transform them to reduced matrices.

5 Reduced Matrices

Now we start to form matrix functions. For this purpose, with reduced spectra in hand, we turn here to their simple transformation – reduced matrices, being values of such functions. These matrices comes from reduced spectra by easy isomorphic enumeration of their domains:

Definition 5.1
1) *We name as a matrix reduced to an ordinal* χ *every relation* M *satisfying the following formula denoted by* $\mu(M, \chi)$ *:*

$$(M \textit{ is a function}) \wedge (dom(M) \textit{ is an ordinal}) \wedge rng(M) \subseteq B_{\chi} \quad .$$

2) *Let* M *be a matrix and* $M_1 \subset k \times B$. *We name as a superimposition of* M *onto* M_1 *a function* f *satisfying the following formula denoted by* $f : M \Rightarrow M_1$ *:*

$$(f \quad is \ an \ order \ isomorphism \ of \ dom(M) \ onto \ \ dom(M_1)) \wedge$$

$$\wedge \forall \alpha, \alpha' \forall \Delta, \Delta' (f(\alpha) = \alpha' \wedge (\alpha, \Delta) \in M \wedge (\alpha', \Delta') \in M_1 \longrightarrow$$

$$\longrightarrow \Delta = \Delta').$$

If the superimposition exists then we say that M *is superimposed onto* M_1 *and write* $M \Rightarrow M_1$.
3) *If the matrix* M *superimposes onto the spectrum* $\widetilde{\mathbf{S}}_n^{\sin \triangleleft \alpha} \overline{\overline{\lceil}} \chi$ *then* M *is named the matrix of this spectrum on* α, *or the subinaccessibly universal matrix of the level* n *reduced to* χ *on* α.
4) *In this case if* $(\alpha', \Delta) \in \widetilde{\mathbf{S}}_n^{\sin \triangleleft \alpha} \overline{\overline{\lceil}} \chi$, *then* α' *is named the jump cardinal of the matrix* M, *while* Δ *is named its Boolean value on* α.
5) *In this case the cardinal* α *is named the carrier of the matrix* M.

By this definition if $M \Rightarrow \widetilde{\mathbf{S}}_n^{\sin\lhd\alpha}\overline{\overline{\lceil}}\chi$, then

$$rng(M) = rng(\widetilde{\mathbf{S}}_n^{\sin\lhd\alpha}\overline{\overline{\lceil}}\chi),$$

so from here we shall consider the Boolean spectrum on α also as $rng(M)$.

Matrices introduced above can be named one-dimensional; following this definition it is possible to define multi-dimensional matrices superimposed onto multi-dimensional reduced spectra of the same dimension (see comment 4)) with a view to a finer analysis of propositions of our language. Matrices of this multi-dimensional kind were used by the author for a long time as the main tool of investigations of subinaccessibility.
In what follows carriers α and reducing cardinals χ will be SIN_{n-2}-cardinals, $\chi < \alpha \leq k$ (if some other case is not considered).

From this definition and lemmas 4.4, 4.5 there come the following two lemmas:

Lemma 5.2
The formulas

$$f : M \Rightarrow \widetilde{\mathbf{S}}_n^{\sin\lhd\alpha}\overline{\overline{\lceil}}\chi \quad , \; M \Rightarrow \widetilde{\mathbf{S}}_n^{\sin\lhd\alpha}\overline{\overline{\lceil}}\chi$$

belong respectively to Π_n, Σ_{n+1} *for* $\alpha = k$ *and to* Δ_1 *for* $\alpha < k$.

Lemma 5.3
Let

$$M \Rightarrow \widetilde{\mathbf{S}}_n^{\sin\lhd\alpha}\overline{\overline{\lceil}}\chi,$$

then

1) $\left\| \widetilde{u}_n^{\sin \lhd \alpha}(\underline{l}) \right\| \lceil \chi = \sum rng(M)$;

2) $OT(M) = dom(M) \leq Od(M) < \chi^+$.

The last statement shows that now reduced matrices can be compared (in the sense of Gödel function Od) *within* L_{χ^+} *only* and this circumstance will make possible to define matrix functions with required properties.

The main role further is played by matrices and spectra reduced to complete cardinals; their existence comes out from lemma 2.5 1) (for $\widetilde{u}_n^{\sin}$, k as φ, α_1 respectively):

Definition 5.4

We name as a complete ordinal of level n *every ordinal* χ *such that*

$$\exists x\ \widetilde{u}_{n-1}^{\sin}(x, \underline{l}) \dashv\vdash \exists x \lhd^{\underline{l}} \chi\ \widetilde{u}_{n-1}^{\sin}(x, \underline{l}).$$

The least of these ordinals is denoted by χ^*, *while the value* $\left\| \widetilde{u}_n^{\sin}(\underline{l}) \right\|$ *is denoted by* A^* .

Lemma 5.5

1) $\chi^* = \sup dom \left(\widetilde{\mathbf{S}}_n^{\sin} \right) = \sup dom\,(\mathbf{S}_n) < k$;

2) $SIN_{n-1}(\chi^*)$, $\chi^* = \omega_{\chi^*}$;

3) $OT(\chi^* \cap SIN_{n-1}) = OT(dom\left(\widetilde{\mathbf{S}}_n^{\sin}\right)) = OT(dom\,(\mathbf{S}_n)) = \chi^*$;

4) $\widetilde{\mathbf{S}}_n^{\sin} = \widetilde{\mathbf{S}}_n^{\sin \lhd \chi^*} = \widetilde{\mathbf{S}}_n^{\sin} \overline{\overline{\lceil}} \chi^*$; $\mathbf{S}_n = \mathbf{S}_n^{\lhd \chi^*} = \mathbf{S}_n \overline{\overline{\lceil}} \chi^*$; $A^* = A^* \lceil \chi^*$;

and similarly for the reduction of these spectra to every SIN_{n-1}-*cardinal* $\geq \chi^*$.

Proof of 1) follows from lemmas 3.10 2), 2.7, 1.1; statement 2) follows from lemmas 3.10 2), 3.4 1), 3.5 (for the level $n-1$) when $\alpha_1 = k$. Then 3), 4) are deduced by the splitting method of the proof of lemma 4.6. ⊣
One can show that lemma 5.5 2) is best possible and $\chi^* \notin SIN_n$.

From this lemma it follows one more important property of the complete cardinal reinforcing lemma 3.2 (for $\alpha_1 = k, \alpha = \chi^*$) because here χ^* is only the SIN_{n-1}-cardinal:

Lemma 5.6
Let $\forall x\ \varphi$ *be a* $\Pi_n^{\dashv\vdash}$*-proposition,* $\varphi \in \Sigma_{n-1}^{\dashv\vdash}$, *then:*

1) *if* φ *contains individual constants only from* $L_{\chi^*}^{B_{\chi^*}}$, *then for every* $\mathfrak{M}$*-generic function* l

$$L_k[l] \vDash \left(\forall x \lhd^{\underline{l}} \chi^*\ \varphi \longleftrightarrow \forall x\ \varphi\right)\ ;$$

2) *if* φ *contains individual constants* $\lhd\, \chi^*$ *only from* L_k *and does not contain* $\underline{l}$, *then*

$$L_k \vDash (\forall x \lhd \chi^*\ \varphi \longleftrightarrow \forall x\ \varphi)\ ;$$

3) *let* $\omega_0^* = \sup dom\left(\widetilde{\mathbf{S}}_n^{\sin}\overline{\overline{\lceil}}(\omega_0 + 1)\right)$ *and* φ *does not contain individual constants and* $\underline{l}$, *then*

$$L_k \vDash (\forall x \lhd \omega_0^*\ \varphi \longleftrightarrow \forall x\ \varphi)\ .$$

Proofs of 1) - 3) are analogous and come to the fact that all jump cardinals of the proposition $\exists x\ \neg\varphi$ (if they exist) are less than χ^*, while for 3) even than ω_0^*; so, they can be demonstrated first for 2).
Beforehand the following remark should be done: for every $x \in L_k$, $m \geq 2$, $\alpha \in SIN_m$ and $\mathfrak{M}$-generic l $\quad x \lhd \alpha \longleftrightarrow x \lhd^l \alpha$.

Therefore the restriction $x \lhd \alpha$ should be considered as $Od(x) < \alpha$ over L_k and as $Od^l(x) < \alpha$ over $L_k[l]$.
Let us consider the proposition $\forall x \ \ \varphi(x, \alpha_0)$, $\varphi \in \Sigma_{n-1}$, having only one individual ordinal constant $\alpha_0 \lhd \chi^*$ (for more clearness) and let $\alpha_0 \in dom\left(\widetilde{\mathbf{S}}_n^{\sin}\right)$ (otherwise we can use lemma 5.5. 3)).
First the constant α_0 must be fixed as in the proof of lemma 4.6 by some $\mathfrak{M}$-generic $l \overset{*}{\in} \widetilde{\triangle}_n^{\sin}(\alpha_0)$. Let us assume that

$$L_k \models \exists x \neg \varphi(x, \alpha_0),$$

then

$$L_k[l] \models \exists \alpha \ \ \varphi_1(\alpha, l)$$

where $\varphi_1(\alpha, \underline{l})$ is the following Π_{n-1}-formula:

$$SIN_{n-1}(\alpha) \wedge \exists \alpha' < \alpha \left(\underline{l} \overset{*}{\in} \widetilde{\triangle}_n^{\sin \lhd \alpha}(\alpha') \wedge \exists x \lhd \alpha \neg \varphi^L(x, \alpha') \right).$$

The proposition $\varphi_2(\underline{l}) = \exists \alpha \ \ \varphi_1(\alpha, \underline{l})$ already has no individual constants and $l \overset{*}{\in} \triangle_{\varphi_2}(\alpha)$ for some its jump ordinal α. By lemmas 2.7, 5.5 1) $\alpha < \chi^*$ and for some $\alpha' < \alpha$

$$L_k[l] \models \left(l \overset{*}{\in} \widetilde{\triangle}_n^{\sin \lhd \alpha}(\alpha') \wedge \exists x \lhd \alpha \neg \varphi^L(x, \alpha') \right).$$

Since $\alpha \in SIN_{n-1}$ and $\alpha' < \alpha$ we can drop here the restriction $\lhd^l \alpha$ and hence $l \overset{*}{\in} \widetilde{\triangle}_n^{\sin}(\alpha')$. It implies $\alpha' = \alpha_0$ and

$$L_k[l] \models \exists x \lhd \alpha \neg \varphi^L(x, \alpha_0)$$

and, at last,

$$L_k \models \exists x \lhd \chi^* \neg \varphi(x, \alpha_0).$$

Turning to 3) let us assume that $L_k \models \exists x \neg \varphi(x)$, then for every

$\mathfrak{M}$-generic l

$$L_k[l] \models \exists\alpha \quad \varphi_1^L(\alpha)$$

where

$$\varphi_1(\alpha) = \exists x \lhd \alpha \neg\varphi(x).$$

From lemma 1.4 it is clear that

$$\left\| \exists\alpha \quad \varphi_1^L(\alpha) \right\| = 1$$

and for every α

$$\left\| \varphi_1^L(\alpha) \right\| \in \{0;1\}\,.$$

Let $\mathfrak{n}$ be the Gödel number of the formula $\exists\alpha \quad \varphi_1^L(\alpha)$ and $p_\mathfrak{n} = \{(\omega_0, \mathfrak{n})\}$. From here and definitions 2.6, 3.9 one can see that

$$p_\mathfrak{n} \le \|\exists\alpha \quad \widetilde{u}_{n-1}(\alpha, \underline{l})\|$$

and for every α

$$p_\mathfrak{n} \cdot \|\widetilde{u}_{n-1}(\alpha, \underline{l})\| = 0 \quad \text{or} \quad p_\mathfrak{n} \le \|\widetilde{u}_{n-1}(\alpha, \underline{l})\| \;.$$

These statements are preserved under reducing to $\chi = \omega_0 + 1$ the values $\|\widetilde{u}_{n-1}(\alpha, \underline{l})\|$. By definition 4.1 it implies the existence of some jump cardinal $\alpha_\mathfrak{n}$ such that

$$p_\mathfrak{n} \cdot \widetilde{\triangle}_n^{\sin}(\alpha_\mathfrak{n})\overline{\lceil}(\omega_0 + 1) > 0.$$

Hence

$$\alpha_\mathfrak{n} \in dom\left(\widetilde{\mathbf{S}}_n^{\sin}\overline{\overline{\lceil}}(\omega_0+1)\right) \text{ and } \alpha_\mathfrak{n} < \omega_0^*.$$

After that by definitions 3.9, 2.6

$$L_k[l] \models \exists\alpha' < \alpha_\mathfrak{n} \quad \varphi_1^L(\alpha')$$

and, at last,

$$L_k \models \exists x \lhd \omega_0^* \neg\varphi(x).$$

$\dashv$

Using here the arguments from the proof of 2) we can admit individual constants $\in dom\left(\widetilde{\mathbf{S}}_n^{\sin}\overline{\overline{\lceil}}(\omega_0+1)\right)$ in $\forall x\varphi$, but it is not necessary in what follows.

Further the very special role is played by the so called *singular matrices*:

Definition 5.7

We denote by $\sigma(\chi,\alpha)$ *the conjunction of the following formulas:*

1) $SIN_{n-2}(\alpha)\wedge(\chi$ *is a limit cardinal* $<\alpha)$ *;*

2) $OT(SIN_{n-1}^{<\alpha})=\alpha$ *;*

3) $\sup dom\left(\widetilde{\mathbf{S}}_n^{\sin\lhd\alpha}\overline{\overline{\lceil}}\chi\right)=\alpha$.

And let $\sigma(\chi,\alpha,M)$ *denote the formula*

$$\sigma(\chi,\alpha)\wedge(M\Rightarrow\widetilde{\mathbf{S}}_n^{\sin\lhd\alpha}\overline{\overline{\lceil}}\chi).$$

The matrix M *and the spectrum* $\widetilde{\mathbf{S}}_n^{\sin\lhd\alpha}\overline{\overline{\lceil}}\chi$ *reduced to* χ *are named singular on a carrier* α *(on an interval* $[\alpha_1,\alpha_2[$ *) iff* $\sigma(\chi,\alpha,M)$ *is fulfilled (for some* $\alpha\in[\alpha_1,\alpha_2[$ *).*
The symbol S *is used for the common notation of singular matrices.*

In what follows all matrices *will be reduced to certain cardinal* χ *and singular on their carriers under consideration*; all reasoning will be conducted in L_k (or in $\mathfrak{M}$ if the context does not point to the opposite case).

Definition 5.7, lemma 5.2 imply easily

Lemma 5.8
1) *The formulas* $\sigma(\chi,\alpha)$, $\sigma(\chi,\alpha,M)$ *belong to* Π_{n-2}.
2) *If* $\sigma(\chi,\alpha)$, *then*

$$\widetilde{A}_n^{\sin\lhd\alpha}(\chi)\lceil\chi < \sum rng\left(\widetilde{\mathbf{S}}_n^{\sin\lhd\alpha\overline{\overline{\lceil}}}\chi\right) = \left\|\widetilde{u}_n^{\sin\lhd\alpha}(\underline{l})\right\|\lceil\chi.$$

Due to lemma 5.8 2) it is possible to introduce the following important cardinals:

Definition 5.9
Let $\sigma(\chi,\alpha,S)$ *fulfills, then we name as jump cardinal and prejump cardinal after* χ *of the matrix* S *on the carrier* α, *or, briefly, of the cardinal* α, *the following cardinals respectively:*

$$\alpha_\chi^{\downarrow} = \min\{\alpha' \in]\chi,\alpha[\ : \ \widetilde{A}_n^{\sin\lhd\alpha}(\chi)\lceil\chi < \widetilde{A}_n^{\sin\lhd\alpha}(\alpha')\lceil\chi \wedge$$
$$\wedge SIN_{n-1}^{<\alpha}(\alpha')\};$$
$$\alpha_\chi^{\Downarrow} = \sup\{\alpha' < \alpha_\chi^{\downarrow} \ : \ \widetilde{A}_n^{\sin\lhd\alpha}(\chi)\lceil\chi = \widetilde{A}_n^{\sin\lhd\alpha}(\alpha')\lceil\chi \wedge$$
$$\wedge SIN_{n-1}^{<\alpha}(\alpha')\}.$$

In this notation and everywhere further the index χ can be omitted if it is arbitrary or can be restored from the context.

Lemma 5.10
Let $\sigma(\chi,\alpha,S)$ *fulfills, then the cardinals* $\alpha_\chi^{\downarrow}$, $\alpha_\chi^{\Downarrow}$ *do exist and*

1) $\alpha_\chi^{\downarrow} = \min\left\{\alpha' > \chi : \alpha' \in dom\left(\widetilde{\mathbf{S}}_n^{\sin\lhd\alpha\overline{\overline{\lceil}}}\chi\right)\right\}$;

2) $\alpha_\chi^{\Downarrow} < \alpha_\chi^{\downarrow} < \alpha$; $]\alpha_\chi^{\Downarrow},\alpha_\chi^{\downarrow}[\cap SIN_{n-1}^{<\alpha} = \varnothing$;

3) $\alpha_\chi^{\downarrow}, \alpha_\chi^{\Downarrow} \in SIN_{n-2}$.

Proof of 1), 2) follows from the definitions and lemma 3.4, while 3) – from lemmas 3.6, 3.8 2) (for $k, \alpha, n-2$ as α_1, α_2, n respectively). ⊣

The following lemma along with lemma 4.6 about spectrum type constitutes the main tools of treating matrix functions and presents the important informative property of reduced matrices and corresponding spectra.
In the foregoing when considering reduced spectra we have mainly used in our reasoning their first projections, that is ordinal spectra. As for second projections of reduced Boolean spectra or reduced matrices – these ones have played an auxiliary role, serving as an instrument of proving lemmas 4.6, 5.5, 5.6 and others. However, second projections of Boolean spectra or reduced matrices have the following important characteristic property: they contain information about parts of the universe bounded by their jump cardinals on their carriers, so that when spectrum matrix is transposed from one carrier to another the properties of such parts of the universe are preserved. More precisely:

Lemma 5.11 (About matrix informativeness)
Let S *be a matrix reduced to* χ *on carriers* $\alpha_1, \alpha_2 > \chi$ *and superimposed on spectra*

$$S \Rightarrow \widetilde{\mathbf{S}}_n^{\sin \lhd \alpha_1} \overline{\overline{\lceil}} \chi, \quad S \Rightarrow \widetilde{\mathbf{S}}_n^{\sin \lhd \alpha_2} \overline{\overline{\lceil}} \chi$$

on these carriers and

$$\overrightarrow{a_1} = (\alpha_{10}, \alpha_{11}, ..., \alpha_{1m}), \quad \overrightarrow{a_2} = (\alpha_{20}, \alpha_{21}, ..., \alpha_{2m})$$

be trains of cardinals from ordinal spectra

$$dom\left(\widetilde{\mathbf{S}}_n^{\sin\lhd\alpha_1}\overline{\overline{\lceil}}\chi\right), \quad dom\left(\widetilde{\mathbf{S}}_n^{\sin\lhd\alpha_2}\overline{\overline{\lceil}}\chi\right)$$

on these carriers which correspond to the same Boolean values:

$$\widetilde{\mathbf{S}}_n^{\sin\lhd\alpha_1}\overline{\overline{\lceil}}\chi(\alpha_{1i}) = \widetilde{\mathbf{S}}_n^{\sin\lhd\alpha_2}\overline{\overline{\lceil}}\chi(\alpha_{2i}), \quad i = \overline{0, m}.$$

Let, at last, $\psi(x_1, x_2, ..., x_m, \underline{l})$ *be an arbitrary formula of an arbitrary level with free variables* x_1, x_2,..., x_m *without individual constants. Then*

$$\psi^{\lhd\alpha_{10}}(\alpha_{11}, ..., \alpha_{1m}, \underline{l}) \dashv\vdash \psi^{\lhd\alpha_{20}}(\alpha_{21}, ..., \alpha_{2m}, \underline{l}).$$

Proof is conducted by the methods analogous to those used in the proof of lemma 4.6. Let us consider for more transparency the case when the formula ψ contains no free variables and $\underline{l}$ and $\overrightarrow{a_1}$, $\overrightarrow{a_2}$ consist only of the jump cardinals after χ on α_1, α_2 respectively:

$$\alpha_{10} = \alpha_{1\chi}^{\downarrow}, \ \alpha_{20} = \alpha_{2\chi}^{\downarrow}$$

because just this case is needed further. Let α_{11}, α_{21} be the succeeding α_{10}, α_{20} jump cardinals in the reduced ordinal spectra considered respectively on the carriers α_1, α_2. The idea of the proof is the following:

The satisfiability of the proposition $\psi^{\lhd\alpha_{10}}$ means that the value

$$\Delta = \widetilde{\mathbf{S}}_n^{\sin\lhd\alpha_1}\overline{\overline{\lceil}}\chi(\alpha_{11})$$

contains the corresponding condition $p \in P_\chi$ in which it is "encoded". When S is carried over from the carrier α_1 to the carrier α_2 the value

$$\Delta = \widetilde{\mathbf{S}}_n^{\sin\lhd\alpha_2}\overline{\overline{\lceil}}\chi(\alpha_{21})$$

still contains p and it means that $\psi^{\lhd\alpha_{20}}$ is fulfilled. But it is more convenient to use instead of p some $\mathfrak{M}$-generic function l including p. The lower index χ in the notation will be dropped.

So, let Δ be the Boolean value of the spectrum $\widetilde{\mathbf{S}}_n^{\sin \lhd\alpha_1}\overline{\overline{\lceil}}\chi$ corresponding to $\alpha_{10} = \alpha_1^{\downarrow}$, that is

$$\Delta = \widetilde{\Delta}_n^{\sin \lhd\alpha_1}(\alpha_{10})\overline{\lceil}\chi,$$

and let some $\mathfrak{M}$-generic function l holds this Δ fixed, that is $l \overset{*}{\in} \Delta$, and hence it holds fixed α_{10} when talking about the carrier α_1 and holds fixed α_{20} when talking about α_2. By definition 4.1

$$l \overset{*}{\in} \left\|\widetilde{u}_{n-1}^{\lhd\alpha_1}(\alpha_{10}, \underline{l})\right\| \lceil\chi \; ; \tag{5.1}$$

$$l \overset{*}{\notin} \left\|\exists\alpha' < \alpha_{10} \left(SIN_{n-1}^{\lhd\alpha_1}(\alpha') \wedge \widetilde{u}_{n-1}^{\lhd\alpha_1}(\alpha', \underline{l})\right)\right\| \lceil\chi \; . \tag{5.2}$$

Since $\alpha_{10} \in SIN_{n-1}^{<\alpha_1}$ the restriction $\lhd \alpha_1$ in (5.2) can be replaced with the restriction $\lhd \alpha_{10}$. After dropping the reduction to χ in (5.1), (5.2) it comes

$$l_1 \overset{*}{\in} \left\|\widetilde{u}_{n-1}^{\lhd\alpha_1}(\alpha_{10}, \underline{l})\right\| \; ;$$

$$l_1 \overset{*}{\notin} \left\|\exists\alpha' < \alpha_{10} \left(SIN_{n-1}^{\lhd\alpha_{10}}(\alpha') \wedge \widetilde{u}_{n-1}^{\lhd\alpha_{10}}(\alpha', \underline{l})\right)\right\| \; ,$$

where l_1 is some $\mathfrak{M}$-generic function coinciding with l on χ; but for some shortness let us use the present symbol l. Then it means that

$$L_k[l] \vDash \widetilde{u}_{n-1}^{\lhd\alpha_1}(\alpha_{10}, l) \wedge \neg\exists\alpha' < \alpha_{10} \left(SIN_{n-1}^{\lhd\alpha_{10}}(\alpha') \wedge \widetilde{u}_{n-1}^{\lhd\alpha_{10}}(\alpha', l)\right) \; . \tag{5.3}$$

Thus the cardinal α_{10} is defined in $L_k[l]$ through l , α_1 and similarly α_{20} is defined through l, α_2 in $L_k[l]$ also. Now let us consider the proposition $\varphi(\underline{l}) = \exists\alpha \ \ \psi_1(\alpha, \underline{l})$ where $\psi_1(\alpha, \underline{l})$ is the following Π_{n-1}-formula:

$$SIN_{n-1}(\alpha) \wedge \widetilde{u}_{n-1}(\alpha, \underline{l}) \wedge \neg\exists\alpha' < \alpha \left(SIN_{n-1}^{<\alpha}(\alpha') \wedge \widetilde{u}_{n-1}^{\triangleleft\alpha}(\alpha', \underline{l})\right) \wedge \psi^{\triangleleft\alpha} .$$

From (5.3) it is clear that

$$L_k[l] \vDash \psi^{\triangleleft\alpha_{10}} \longleftrightarrow L_k[l] \vDash \exists\alpha < \alpha_1 \ \psi_1^{\triangleleft\alpha_1}(\alpha, l) \ . \qquad (5.4)$$

After that the argument from the proof of lemma 2.7 should be repeated: the function l must be replaced with the function l_0 as in (2.1) and at the same time the formula φ must be transformed to the formula φ_2 just as it was done in this proof, that is by replacing its subformulas of the kind $\underline{l}(t_1) = t_2$ with the subformulas (2.2) and so on. Let $\mathfrak{n}$ be the Gödel number of φ_2 and $l_0(\omega_0) = \mathfrak{n}$. As a result we turn from φ to the spectral universal formula and (5.4) implies

$$L_k[l] \vDash \psi^{\triangleleft\alpha_{10}} \longleftrightarrow L_k[l_0] \vDash \exists x \triangleleft \alpha_1 \ u_{n-1}^{\triangleleft\alpha_1}(x, l_0) \ .$$

Using here $\widetilde{u}_{n-1}^{\text{sin}}$ instead of u_{n-1} one can see from definition 5.7 that

$$L_k[l] \vDash \psi^{\triangleleft\alpha_{10}} \longleftrightarrow L_k[l_0] \vDash \exists x \triangleleft \alpha_1 \widetilde{u}_{n-1}^{\text{sin}\,\triangleleft\alpha_1} \ (x, l_0) \ . \qquad (5.5)$$

Analyzing the construction of the formula $\widetilde{u}_n^{\sin}$ it is not hard to derive from (5.5), (5.3) that

$$L_k[l] \models \psi^{\triangleleft\alpha_{10}} \longleftrightarrow l_0 \overset{*}{\in} \widetilde{\Delta}_n^{\sin\triangleleft\alpha_1}(\alpha_{11}) \ ;$$

let us remind that here α_{11} is the successor of α_{10} in the reduced ordinal spectrum on the carrier α_1. In this reasoning the values $l_0(\alpha)$ for $\alpha \geq \chi$ were not used and that is why

$$L_k[l] \models \psi^{\triangleleft\alpha_{10}} \longleftrightarrow l_0 \overset{*}{\in} \widetilde{\Delta}_n^{\sin\triangleleft\alpha_1}(\alpha_{11})\overline{\lceil}\chi \quad . \tag{5.6}$$

The matrix S is superimposed on $\widetilde{\mathbf{S}}_n^{\sin\triangleleft\alpha_2}\overline{\overline{\lceil}}\chi$ and hence

$$L_k[l] \models \psi^{\triangleleft\alpha_{10}} \longleftrightarrow l_0 \overset{*}{\in} \widetilde{\Delta}_n^{\sin\triangleleft\alpha_2}(\alpha_{21})\overline{\lceil}\chi \ . \tag{5.7}$$

This reasoning up to (5.6) but with $\alpha_1, \alpha_{10}, \alpha_{11}$ replaced respectively with $\alpha_2, \alpha_{20}, \alpha_{21}$ establishes that

$$L_k[l] \models \psi^{\triangleleft\alpha_{20}} \longleftrightarrow l_0 \overset{*}{\in} \widetilde{\Delta}_n^{\sin\triangleleft\alpha_2}(\alpha_{21})\overline{\lceil}\chi \ .$$

It remains to compare this equivalence with (5.7).
In the case when ψ contains free variables $x_1 ..., x_m$ and $\underline{l}$ one should carry out the analogous reasoning considering several $\mathfrak{M}$-generic functions

$$l_i \overset{*}{\in} \widetilde{\Delta}_n^{\sin\triangleleft\alpha_1}(\alpha_{1i})\overline{\lceil}\chi, \quad i = \overline{0, m},$$

and treating their direct product as in the proof of lemma 4.6. ⊣

Changing this reasoning slightly it is not hard to conduct it for the prejump cardinals $\alpha_{10} = \alpha^{\Downarrow}_{1\chi}$, $\alpha_{20} = \alpha^{\Downarrow}_{2\chi}$.

The basic instruments of the proof of main theorem are matrix functions that are sequences of reduced singular matrices of the special kind. The following lemma makes it possible to build such functions:

Lemma 5.12

$$\forall \chi \forall \alpha_0 ((\chi \textit{ is a limit cardinal } > \omega_0) \to \exists \alpha_1 > \alpha_0 \ \sigma(\chi, \alpha_1)).$$

Proof. This Π_n-proposition does not contain any individual constants or $\underline{l}$ and therefore by lemma 5.6 3) it is sufficient to prove that it is fulfilled when the variables χ, α_0 are bounded by the cardinal ω_0^*.
So, let χ, $\alpha_0 < \omega_0^*$; let us consider

$$\alpha_1 = \sup dom \left(\widetilde{\mathbf{S}}^{\mathrm{sin}\,\overline{\overline{\ }}}_n \lceil \chi \right).$$

One can see that $\alpha_1 \geq \omega_0^*$ and that is why χ, $\alpha_0 < \alpha_1$. From lemmas 4.2, 3.4 (for $\alpha_1 = k$) it comes $\alpha_1 \in SIN_{n-1}$ and hence conditions 1), 3) of definition 5.7 are carried out; condition 2) can be deduced by the method which is the simplified variant of the splitting method of the proof of lemma 4.6. Really, suppose that, on the contrary,

$$\alpha_2 = OT\left(SIN^{<\alpha_1}_{n-1}\right) < \alpha_1 ,$$

then there exists the cardinal

$$\alpha_3 \in [\alpha_2, \alpha_1[\cap dom \left(\widetilde{\mathbf{S}}^{\mathrm{sin}\,\overline{\overline{\ }}}_n \lceil \chi \right)$$

and some $\mathfrak{M}$-generic function $l \overset{*}{\in} \widetilde{\Delta}_n^{\sin}(\alpha_3)\overline{\lceil}\chi$. Now let us consider the proposition $\varphi(\underline{l}) = \exists\alpha \ \ \varphi_1(\alpha,\underline{l})$ where $\varphi_1(\alpha,\underline{l})$ is the Π_{n-1} formula:

$$SIN_{n-1}(\alpha) \wedge$$
$$\wedge \exists \alpha', \alpha'' < \alpha \ \ (\alpha' < \alpha'' \wedge SIN_{n-1}^{<\alpha}(\alpha'') \wedge \widetilde{u}_{n-1}^{\sin \triangleleft \alpha''}(\alpha', \underline{l}) \wedge$$
$$\wedge \neg \exists \alpha''' < \alpha' \ \ \widetilde{u}_{n-1}^{\sin \triangleleft \alpha''}(\alpha''', \underline{l}) \wedge$$
$$\wedge OT\left(\left\{\alpha''' < \alpha'' : SIN_{n-1}^{<\alpha''}(\alpha''')\right\}\right) = \alpha') \ .$$

It is not hard to see that $L_k[l] \vDash \exists\alpha \ \ \varphi_1(\alpha, l)$ and for every α

$$L_k[l] \vDash \varphi_1(\alpha, l) \longleftrightarrow SIN_{n-1}(\alpha) \wedge \exists \alpha'' < \alpha(SIN_{n-1}(\alpha'') \wedge$$
$$\wedge OT\left(\alpha'' \cap SIN_{n-1}\right) = \alpha_3).$$

Hence $dom\left(\mathbf{S}_\varphi \overline{\overline{\lceil}} \chi\right)$ contains the cardinal $> \alpha_1$ and with the help of lemmas 2.7, 3.10 one can see that $dom\left(\widetilde{\mathbf{S}}_n^{\sin} \overline{\overline{\lceil}} \chi\right)$ contains such cardinals also in contradiction with the assumption. $\dashv$

The building of matrix functions relies on the following enumeration (in L_k) of subinaccessible cardinals:

Definition 5.13

Let $\alpha_1 \le k$.

By the recursion on $\tau < \alpha_1$ *we define the function* $\gamma_f^{<\alpha_1} = (\gamma_\tau^{<\alpha_1})_\tau$ *:*

$$\gamma_0^{<\alpha_1} = 0 \ ; \quad for \quad \tau > 0$$

$$\gamma_\tau^{<\alpha_1} = \min\{\gamma < \alpha_1 : SIN_{n-1}^{<\alpha_1}(\gamma) \wedge \forall \tau' < \tau \quad \gamma_{\tau'}^{<\alpha_1} < \gamma\} \ .$$

The inverse function $\tau_f^{<\alpha_1} = (\tau_\gamma^{<\alpha_1})_\gamma$ *is defined :*

$$\tau = \tau_\gamma^{<\alpha_1} \longleftrightarrow \gamma = \gamma_\tau^{<\alpha_1}.$$

The proof of main theorem consists in creation in L_k of the special matrix function possessing inconsistent properties; this function arises out by a sequential complication of its following simplest form:

Definition 5.14

We name as a matrix function of the level n *below* α_1 *reduced to* χ *the following function* $S_{\chi f}^{<\alpha_1} = (S_{\chi\tau}^{<\alpha_1})_\tau$ *taking values :*

$$S_{\chi\tau}^{<\alpha_1} = \min_{\trianglelefteq}\{S : \exists \alpha < \alpha_1 \left(\gamma_\tau^{<\alpha_1} < \alpha \wedge \sigma^{\triangleleft\alpha_1}(\chi, \alpha, S)\right)\}.$$

So, these values are matrices S reduced to χ and singular on these carriers α.

As usual, if $\alpha_1 < k$, then all functions introduced are named restricted or relativized to α_1; if $\alpha_1 = k$, then all mentionings about α_1 are dropped.
Let us remind that all bounding ordinals α_1 are assumed to be SIN_{n-2}-cardinals or $\alpha_1 = k$.

Lemma 5.15 (About matrix function absoluteness).
Let $\chi < \gamma^{<\alpha_1}_{\tau+1} < \alpha_2 < \alpha_1 \leq k$ *and* $\alpha_2 \in SIN^{<\alpha_1}_{n-2}$, *then:*

1) *functions* $\gamma^{<\alpha_2}_f$, $\gamma^{<\alpha_1}_f$ *coincide on the set* $\{\tau' : \gamma^{<\alpha_2}_{\tau'} \leq \gamma^{<\alpha_1}_{\tau+1}\}$*:*

$$\gamma^{<\alpha_2}_{\tau'} \leq \gamma^{<\alpha_1}_{\tau+1} \longrightarrow \gamma^{<\alpha_2}_{\tau'} = \gamma^{<\alpha_1}_{\tau'};$$

2) *functions* $S^{<\alpha_2}_{\chi f}$, $S^{<\alpha_1}_{\chi f}$ *coincide on the set* $\{\tau' : \chi \leq \gamma^{<\alpha_2}_{\tau'} \leq \gamma^{<\alpha_1}_{\tau}\}$*:*

$$\chi \leq \gamma^{<\alpha_2}_{\tau'} \leq \gamma^{<\alpha_1}_{\tau} \longrightarrow S^{<\alpha_2}_{\chi\tau'} = S^{<\alpha_1}_{\chi\tau'}.$$

Proof is coming from lemma 3.8 1) (where n is replaced with $n-1$) and lemma 5.17 2 (ii) below. ⊣

From lemma 3.3 where n is replaced with $n-1$ it comes

Lemma 5.16
For $\alpha_1 < k$ *functions*

$$\gamma = \gamma^{<\alpha_1}_{\tau}, \quad S = S^{<\alpha_1}_{\chi\tau}$$

are Δ_1*-definable through* χ, α_1.
For $\alpha_1 = k$ *these functions are* Π_{n-1}*-definable,* Δ_n*-definable respectively.*

The following lemma contains the “embryo” of all further reasoning: it establishes that matrix function has the property of $\underline{\prec}$-monotonicity – and further we shall modify this function in order to preserve and to exclude this property simultaneously.
Therefore we shall often return to the idea of this lemma and of its proof in various forms:

Lemma 5.17

1) *The function* $S^{<\alpha_1}_{\chi f}$ *is* $\lessdot$*-monotone, that is for every* $\tau_1,\, \tau_2 \in dom\left(S^{<\alpha_1}_{\chi f}\right)$

$$\tau_1 < \tau_2 \longrightarrow S^{<\alpha_1}_{\chi\tau_1} \lessdot S^{<\alpha_1}_{\chi\tau_2} \ .$$

2) *Let* $\tau \in dom\left(S^{<\alpha_1}_{\chi f}\right)$, *then:*

(i) $\{\tau' : \chi \leq \gamma^{<\alpha_1}_{\tau'} \leq \gamma^{<\alpha_1}_{\tau}\} \subseteq dom\left(S^{<\alpha_1}_{\chi f}\right)$ *;*

(ii) if $\gamma^{<\alpha_1}_{\tau+1}$ *and the matrix* $S^{<\alpha_1}_{\chi\tau}$ *exist, then this matrix possesses a carrier* $\alpha \in \left]\gamma^{<\alpha_1}_{\tau}, \gamma^{<\alpha_1}_{\tau+1}\right[$.

Proof. Statements 1, 2 (i) follow from definition 5.14 immediately. The proof of 2 (ii) represents the typical application of lemma 3.2 about restriction. Since there is the matrix $S = S^{<\alpha_1}_{\chi\tau}$ on some carrier

$$\alpha \in]\gamma^{<\alpha_1}_{\tau}, \alpha_1[,$$

the following Σ_{n-1}-proposition $\varphi(\chi, \gamma^{<\alpha_1}_{\tau}, S)$:

$$\exists\alpha \ \ (\gamma^{<\alpha_1}_{\tau} < \alpha \wedge \sigma(\chi, \alpha, S))$$

is fulfilled below α_1, that is fulfilled $\varphi^{\lhd\alpha_1}(\chi, \gamma^{<\alpha_1}_{\tau}, S)$.

This proposition contains individual constants

$$\chi < \gamma^{<\alpha_1}_{\tau+1}, \quad \gamma^{<\alpha_1}_{\tau} < \gamma^{<\alpha_1}_{\tau+1}, \quad S \lhd \gamma^{<\alpha_1}_{\tau+1}$$

due to lemma 5.3 2), and that is why by lemma 3.2 (where n replaced with $n-1$) the $SIN^{<\alpha_1}_{n-1}$-cardinal $\gamma^{<\alpha_1}_{\tau+1}$ restricts the proposition φ, that is fulfilled the formula

$$\exists\alpha \in \left[\gamma^{<\alpha_1}_{\tau}, \gamma^{<\alpha_1}_{\tau+1}\right[\ \sigma^{\lhd\alpha_1}(\chi, \alpha, S).$$

$\dashv$

From here, lemma 5.12 (for $\chi = \chi^*$) and lemma 5.3 2) (for $\alpha_1 = k$) it comes directly

Lemma 5.18
1) *The unrestricted function* $S_{\chi^* f}$ *is defined on the final segment of* k*:*

$$dom\,(S_{\chi^* f}) = \{\tau : \chi^* \leq \gamma_\tau < k\}\,;$$

2) *this function stabilizes, that is there is an ordinal* $\tau^* > \chi^*$ *such that for every* $\tau \geq \tau^*$ *there exists*

$$S_{\chi^* \tau} = S_{\chi^* \tau^*};$$

therefore for every $\gamma_\tau \geq \chi^*$

$$S_{\chi^* \tau} \trianglelefteq S_{\chi^* \tau^*}.$$

Now let us make an outline of the first approach to the idea of the main theorem proof.
We shall try to obtain the required contradiction on the following way. The lower index χ^* will be dropped for some brevity.
Let us consider the matrix function S_f in its state of stabilizing, that is consider an arbitrary sufficiently large $\tau_0 > \tau^*$, the matrix S_{τ_0} on some carrier $\alpha_0 \in]\gamma_{\tau_0}, \gamma_{\tau_0+1}[$, the prejump cardinal $\alpha^0 = \alpha_0^{\Downarrow}$ and the function $S_f^{<\alpha^0}$.
The reasoning forthcoming will be more convenient and transparent, if the following notion will be introduced:

We shall say, that one, standing on an ordinal $\alpha > \chi^*$*, can detect (or can see) some information or some object below* α*, iff this information or object can be defined by some formula relativized* *to this* α*, and defined just as some ordinal* $< \chi^{*+}$*.*

So, the idea of the matrix function research in its basic form will be mainly the following:

Assume, that some matrix, which is the value of such function, is considered on its carrier. Standing on the prejump cardinal α^0 *after* χ^* *of this matrix on this carrier, one should observe the situation below* α^0 *and, mainly, the behavior of this very function, but in its relativized to* α^0 *form, with a view to detect its properties.*

The information about this function, that one can detect and express as some ordinals $< \chi^{*+}$ – this information can provide certain contradictions.
For instance, it can provide the increasing of the order type of S_{τ_0} due to lemma 4.6, and, hence, can destroy the stabilizing of the whole function S_f in contraction with lemma 5.18.
More explicitly in this context:
By lemma 5.15 2) the function $S_f^{<\alpha^0}$ coincides with S_f on the ordinal τ_0 and also $\underset{\cdot}{<}$-monotone. Let us apply lemma 4.6 to this situation, considering

$$\overline{\delta} = \sup_\tau Od(S_\tau^{<\alpha^0}), \quad \overline{\chi} = \chi^*, \quad \overline{\alpha}_0 = \alpha_0^{\downarrow}, \quad \overline{\alpha}_1 = \alpha_0.$$

Suppose $\overline{\delta} < \chi^{*+}$, then one can detect $\overline{\delta}$ standing on α^0. But now from definition 5.7 and lemma 5.5 it comes that all conditions of lemma 4.6 are fulfilled and so it implies the contradiction:

$$Od(S_{\tau^*}) \leq \overline{\delta} < OT(S_{\tau_0}) \leq Od(S_{\tau^*}).$$

Hence, in reality $\overline{\delta} = \chi^{*+}$ and the function $S_f^{<\alpha^0}$ is $\underset{\cdot}{<}$-nondecreasing up to χ^{*+}.

It means that the function $(S_\tau)_{\tau<\tau_0}$, being $\trianglelefteq$ -bounded by the ordinal $Od\,(S_{\tau^*}) < \chi^{*+}$, loses this boundedness after its ex-

tension on the set $\{\tau : \gamma_\tau^{<\alpha^0} < \alpha^0\}$.
One can see that it happens because of losing the properties of subinaccessibility of all levels $\geq n$ by all cardinals $\leq \gamma_{\tau_0}$ after their relativization to α^0; at the same time there appears some SIN_{n-1}-cardinals (relativized to α^0) that are not such in the universe (Kiselev [12], [13]).

All these conclusions mean that many important properties of lower levels of the universe do not extend up to the relativizing cardinals, namely, to jump cardinals of reduced matrices, which are values of matrix functions.

In order to prevent this phenomenon we shall introduce special cardinals named disseminators that are extending such properties without distortion.
Extension of this kind sometimes are produced by inaccessible cardinals but for what follows it is needed to widen this notion significantly.

6 Disseminators

Let us remind that all the reasoning is conducted and formulas are interpreted in L_k (or in $\mathfrak{M}$ if, of course, the context does not mean some another case). Also all $\triangleleft$-bounding ordinals belong to SIN_{n-2}.
We shall introduce the notion of disseminator only for the constructive universe for more clearness of reasoning although it can be done without any loss of generality. Further we shall consider classes Σ_m, Π_m of the fixed arbitrary level $m > 3$ (unless otherwise specified).

Definition 6.1

Let

$$0 < \alpha < \alpha_1 \leq k, \quad X \subseteq \alpha, \quad X \neq \varnothing.$$

The ordinal α *is named the disseminator of the level* m *with the data base* X *below* α_1 *iff for every* $\mathfrak{n} \in \omega_0$ *and train* $\overrightarrow{a}$ *of ordinals* $\in X$

$$U_m^{\Sigma \triangleleft \alpha_1}(\mathfrak{n}, \overrightarrow{a}) \longrightarrow U_m^{\Sigma \triangleleft \alpha}(\mathfrak{n}, \overrightarrow{a}) \quad .$$

The formula defining in L_k *the set of all disseminators with the data base* X *of the level* m *below* α_1 *is denoted by* $SIN_m^{<\alpha_1}[X](\alpha)$*; this set itself is also denoted by* $SIN_m^{<\alpha_1}[X]$*, while its disseminators – by the common symbol* δ^X *or, briefly, by* δ*, pointing to* α_1 *in the context.*

Let us remind that the symbol U_m^{Σ} here denotes the Σ_m-formula which is universal for the class of Σ_m-formulas but without any occurrences of the constant $\underline{l}$ (see remark after lemma 2.5).
It is possible to obtain the definition of the disseminator notion in more wide sense if occurrences of the constant $\underline{l}$ are allowed:

for every train $\overrightarrow{a}$ *of ordinals* $\in X$

$$\left\| u_m^{\Sigma \lhd \alpha_1}(\overrightarrow{a}, \underline{l}) \right\| \leq \left\| u_m^{\Sigma \lhd \alpha}(\overrightarrow{a}, \underline{l}) \right\|,$$

and all the following reasoning can be carried over in this case.

The following two lemmas are quite analogous to lemmas 3.3, 3.2 and can be proved in the same way:

Lemma 6.2
The formula $SIN_m^{<\alpha_1}[X](\alpha)$ *belongs to the class* Π_m *for* $\alpha_1 = k$ *and to the class* Δ_1 *for* $\alpha_1 < k$.

The following obvious lemma justifies the term “disseminator” since it shows that such an ordinal really extend Π_m-properties (containing constants from its base) from lower levels of the universe up to relativizing cardinals:

Lemma 6.3 (About extending)
Let

$$X \subseteq \alpha, \quad \alpha < \alpha_1, \quad \alpha \in SIN_m^{<\alpha_1}[X]$$

and a proposition $\forall x\ \varphi(x, \overrightarrow{a})$ *has a train* $\overrightarrow{a}$ *of constants* $\in X$, $\varphi \in \Sigma_{m-1}$, *then :*

$$\forall x \lhd \alpha\ \varphi^{\lhd \alpha}(x, \overrightarrow{a}) \longrightarrow \forall x \lhd \alpha_1\ \varphi^{\lhd \alpha_1}(x, \overrightarrow{a}) .$$

In this case we shall say, as above, that below α_1 *the ordinal* α *extends or prolongs* $\forall x\ \varphi$ *up to* α_1.
Considering the same in the inverted form for $\varphi \in \Pi_{m-1}$:

$$\exists x \lhd \alpha_1\ \varphi^{\lhd \alpha_1}(x, \overrightarrow{a}) \longrightarrow \exists x \lhd \alpha\ \varphi^{\lhd \alpha}(x, \overrightarrow{a}) ,$$

we shall say that below α_1 *the ordinal* α *restricts or relativizes the proposition* $\exists x\ \varphi$. $\dashv$

Evidently, the class of disseminators $\alpha \in SIN_m^{<\alpha_1}[X]$ in *wide sense* below α_1 of level m with bases $X = \alpha$ coincides with $SIN_m^{<\alpha_1}$.

The properties of disseminators become more transparent if, in addition, they possess the properties of subinaccessibility of lower levels:

Lemma 6.4

Let

$$X \subseteq \alpha, \quad \alpha < \alpha_1, \quad \alpha \in SIN_{m-1}^{<\alpha_1},$$

then the following statements are equivalent:

1) $\alpha \in SIN_m^{<\alpha_1}[X]$ *;*

2) *for every* $\mathfrak{n} \in \omega_0$ *and every train* $\overrightarrow{a}$ *of constants* $\in X$

$$L_k \models \exists x \lhd \alpha_1\ U_{m-1}^{\Pi \lhd \alpha_1}(\mathfrak{n}, x, \overrightarrow{a}) \longleftrightarrow \exists x \lhd \alpha\ U_{m-1}^{\Pi \lhd \alpha_1}(\mathfrak{n}, x, \overrightarrow{a})\ ;$$

3) *for every* Π_{m-1}*-formula* $\varphi(x, \overrightarrow{a})$ *and every train* $\overrightarrow{a}$ *of constants* $\in X$

$$L_k \models \exists x \lhd \alpha_1\ \varphi^{\lhd \alpha_1}(x, \overrightarrow{a}) \longleftrightarrow \exists x \lhd \alpha\ \varphi^{\lhd \alpha_1}(x, \overrightarrow{a})\ .$$

$\dashv$

This lemma advances the following important

Definition 6.5

1) *The minimal disseminator of the class*

$$SIN_m^{<\alpha_1}[X] \cap SIN_{m-1}^{<\alpha_1}$$

is named the generating disseminator with the data base X *below* α_1 *and is denoted by the general symbol* $\check{\delta}^X$, *or* $\check{\delta}$ *or* δ*;*
2) *without this condition of minimality, disseminators of this class are named floating disseminators and are denoted by the general symbol* $\widetilde{\delta}^X$ *or, briefly, by* $\widetilde{\delta}$ *or* δ, *pointing to* α_1 *and to their other attributes in the context.* $\dashv$

As usual, indices m, α_1,X are omitted if they are arbitrary or pointed out by the context.
These terms "generating disseminator" and "floating disseminator" are justified by the following circumstance: the *generating disseminator* is *uniquely* defined through its base X below α_1 as minimal, while the *floating disseminator* may be not, and its value is not specified (it is floating); moreover, the "generating disseminator" term is justified by the following

Lemma 6.6
Let

$$X \subseteq \alpha_0, \quad \alpha_0 < \delta < \alpha_1,$$

δ *be a generating disseminator* $\in SIN_m^{<\alpha_1}[X]$ *and* $\varphi(\alpha, \overrightarrow{a})$ *be a* Σ_{m-1}*-formula with the ordinal variable* α *and the train* $\overrightarrow{a}$ *of constants* $\in X$.
Suppose

$$\forall \alpha \in \,]\alpha_0, \delta[\;\; \varphi^{\lhd\delta}(\alpha, \overrightarrow{a}),$$

then there exists some $\alpha_0' \in [\alpha_0, \delta[$ *such that*

$$\forall \alpha \in \,]\alpha_0', \alpha_1[\;\; \varphi^{\lhd\alpha_1}(\alpha, \overrightarrow{a}).$$

Proof. One should point out here that the variable α under the quantor $\forall$ runs not through all δ, as it was in lemma 6.3, but only through its some final segment $[\alpha_0, \delta[$.
Beforehand the following remark should be done: for arbitrary

ordinals $\alpha \in SIN_e$, $e > 1$, β: $\beta < \alpha \longleftrightarrow \beta \lhd \alpha$.
First let us consider the case when $\alpha_0 \in SIN_{m-1}^{<\alpha_1}$. From the minimality of δ it follows that $\alpha_0 \notin SIN_m^{<\alpha_1}[X]$ and by lemma 6.4 it means that there is Σ_{m-1}-formula $\varphi_1(\alpha, \overrightarrow{a_1})$ with the train $\overrightarrow{a_1}$ of constants $\in X$ such that

$$\forall \alpha < \alpha_0 \ \ \varphi_1^{\lhd \alpha_0}(\alpha, \overrightarrow{a_1}) \ , \tag{6.1}$$

$$\exists \alpha' < \alpha_1 \neg \varphi_1^{\lhd \alpha_1}(\alpha', \overrightarrow{a_1}) \ . \tag{6.2}$$

Since $\alpha_0, \delta \in SIN_{m-1}^{<\alpha_1}$, statement (6.1) is equivalent to

$$\forall \alpha < \alpha_0 \ \ \varphi_1^{\lhd \delta}(\alpha, \overrightarrow{a_1}) \ . \tag{6.3}$$

The cardinal δ is the disseminator and that is why (6.2) implies

$$\exists \alpha' < \delta \neg \varphi_1^{\lhd \delta}(\alpha', \overrightarrow{a_1}) \ .$$

From here and (6.3) it comes that there is the ordinal $\alpha_0' \in [\alpha_0, \delta[$ such that

$$\forall \alpha < \alpha_0' \ \ \varphi_1^{\lhd \delta}(\alpha, \overrightarrow{a}_1) \wedge \neg \varphi_1^{\lhd \delta}(\alpha_0', \overrightarrow{a}_1) \ . \tag{6.4}$$

Now from the condition of the lemma it comes the proposition

$$\forall \alpha', \alpha < \delta \left(\neg \varphi_1^{\lhd \delta}(\alpha', \overrightarrow{a}_1) \wedge \alpha > \alpha' \longrightarrow \varphi^{\lhd \delta}(\alpha, \overrightarrow{a}) \right) \ .$$

It contains constants $\in X$ so by lemma 6.4 3) the disseminator δ extends it up to α_1:

$$\forall \alpha', \alpha < \alpha_1 \left(\neg \varphi_1^{\lhd \alpha_1}(\alpha', \overrightarrow{a}_1) \wedge \alpha > \alpha' \longrightarrow \varphi^{\lhd \alpha_1}(\alpha, \overrightarrow{a}) \right) \ . \qquad (6.5)$$

Since $\delta \in SIN_{m-1}^{<\alpha_1}$ from (6.4) it comes $\neg \varphi_1^{\lhd \alpha_1}(\alpha_0', \overrightarrow{a}_1)$ and therefore (6.5) implies $\forall \alpha \in [\alpha_0', \alpha_1[\ \ \varphi^{\lhd \alpha_1}(\alpha, \overrightarrow{a})$. ⊣

Using this lemma we shall say, as above, that δ *extends or prolongs* the proposition $\forall \alpha \ \ \varphi$ up to α_1, or, in the inverted form, that δ *restricts or relativizes* $\exists \alpha \neg \varphi$ below α_1, pointing out ordinals α_0, α_0' by the context (if it will be necessary).

Corollary 6.7
Let $\delta \in SIN_m^{<\alpha_1}[X]$ *be a generating disseminator, then*

$$\sup(\delta \cap SIN_{m-1}^{<\alpha_1}) = \delta.$$

Proof. Let, on the contrary,

$$\alpha_0 = sup\left(\delta \cap SIN_{m-1}^{<\alpha_1}\right) < \delta,$$

then $\alpha_0 \in SIN_{m-1}^{<\alpha_1}$ and the argument from the previous proof can be applied. The disseminator δ restricts the proposition

$$\exists \alpha > \alpha_0 \ \ SIN_{m-1}(\alpha)$$

that is fulfilled below α_1 since $\delta \in SIN_{m-1}^{<\alpha_1}$. So there exists

$$\alpha \in \,]\alpha_0, \delta[\, \cap SIN_{m-1}^{<\alpha_1}$$

contrary to the assumption. ⊣

Now it is clear that in the proof of lemma 6.6 the condition $\alpha_0 \in SIN_{m-1}^{<\alpha_1}$ should be dropped.

In what follows we shall investigate the mutual disposition of disseminators of one and the same level m below one and the same α_1 but with different data bases. How these bases influence their location? The discussion of such questions is more clear when the data base X is some ordinal ρ (that is the set of all smaller ordinals). In order to point out this case we shall write

$$SIN_m^{<\alpha_1}[<\rho] \quad \textit{instead of} \quad SIN_m^{<\alpha_1}[X]\,.$$

Further all bases ρ are *limit* ordinals (if some another case is not meant).

The following lemma, although obvious, represents, nevertheless, several important technical tools of disseminator analysis:

Lemma 6.8
1) *Let*

$$\delta < \alpha_1, \quad \delta \in SIN_m^{<\alpha_1}[X] \cap SIN_{m-1}^{<\alpha_1}$$

then every $SIN_{m-1}^{<\alpha_1}$*-cardinal in* $[\delta, \alpha_1[$ *is again the disseminator with the same base below* α_1*:*

$$\alpha \in SIN_{m-1}^{<\alpha_1} \wedge \alpha > \delta \longrightarrow \alpha \in SIN_m^{<\alpha_1}[X]\,;$$

2) *the increasing of the base* ρ *implies the nondecreasing of the generating disseminator below* α_1*:*

$$\rho_1 < \rho_2 \longrightarrow \check{\delta}^{\rho_1} \le \check{\delta}^{\rho_2}; \quad \rho_1^+ \le \rho_2 \longrightarrow \check{\delta}^{\rho_1} < \check{\delta}^{\rho_2};$$

3) *the passage to the limit of bases involves the passage to the limit of generating disseminators below* α_1*:*

$$\lim_i \rho_i = \rho \longrightarrow \lim_i \check{\delta}^{\rho_i} = \check{\delta}^{\rho}.$$

Proof. 1). The upper index α_1 will be dropped for shortness. Let us turn to definition 6.1 (or to lemma 6.4) and consider the proposition

$$\exists x \ \ U^{\Pi}_{m-1}(\mathfrak{n}, x, \overrightarrow{a})$$

for an arbitrary $\mathfrak{n} \in \omega_0$ and a train $\overrightarrow{a}$ of constants $\in X$.
Suppose, that

$$\exists x \lhd \alpha_1 \ U^{\Pi}_{m-1}(\mathfrak{n}, x, \overrightarrow{a}).$$

Since δ is the disseminator it implies

$$\exists x \lhd \delta \ U^{\Pi}_{m-1}(\mathfrak{n}, x, \overrightarrow{a})$$

and, thus,

$$\exists x \lhd \alpha \ \ U^{\Pi}_{m-1}(\mathfrak{n}, x, \overrightarrow{a}).$$

Here U^{Π}_{m-1} should be $\lhd$-bounded by α since $\alpha \in SIN_{m-1}$ and so $\alpha \in SIN_m[X]$. Statement 2) comes from definitions and along with 1) implies 3). $\dashv$

To obtain more detailed information about disposition of disseminators it is natural and reasonable to consider matrix carriers.
Further by $\widehat{\rho}$ is denoted the closure of ρ under the pair function.

Definition 6.9

Let $\chi < \alpha < \alpha_1$ and S be a matrix which is reduced to a cardinal χ and is singular on a carrier α.
1) *We name as disseminator for S on α (or as disseminator for this carrier) of the level m with a data base X every disseminator δ below the prejump cardinal $\alpha^{\Downarrow}_{\chi}$ with the same parameters, that is every*

$$\delta \in SIN_m^{<\alpha^{\Downarrow}_{\chi}}[X] \cap SIN_{m-1}^{<\alpha^{\Downarrow}_{\chi}}.$$

In this case we say also that the matrix S on α (or its carrier α) possesses this disseminator (with these parameters).

2) *We call as the* eigendisseminator *of the matrix* S *on* α *of the level* m *every its disseminator of this level with the data base* $\rho = \widehat{\rho_1}$, $\rho_1 = Od(S)$, *and denote it through the general symbol* $\widetilde{\delta}^S$ *or* δ^S, *and its data base* ρ *— through* ρ^S; *if* $\widetilde{\delta}^S$ *is minimal with this base* ρ^S, *then it is called the* generating eigendisseminator *for* S *on* α *and is denoted through* $\check{\delta}^S$.

3) *The matrix* S *is called the disseminator matrix or, more briefly, the* δ*-matrix of the level* m *admissible on the carrier* α *for* $\gamma = \gamma_\tau^{<\alpha_1}$ *below* α_1 *iff it possesses some disseminator* $\delta < \gamma$ *of the level* m *with some base* $\rho \leq \chi^{*+}$ *on this carrier such that* $S \lhd \rho$, $\rho = \widehat{\rho}$ *(also below* α_1*).* [6)]

4) *In what follows by* $Lj^{<\alpha}(\chi)$ *is denoted the* Δ_1*-formula:*

$$\chi < \alpha \wedge SIN_{n-1}^{<\alpha}(\chi) \wedge \sum rng\left(\widetilde{\mathbf{S}}_n^{\sin \lhd \chi}\right) \in B_\chi \wedge \sup dom\left(\widetilde{\mathbf{S}}_n^{\sin \lhd \chi}\right) = \chi\ .$$

and the cardinal χ *with this property is called saturated below* α.
Now let us accept the following important agreement:
everywhere further we shall consider singular matrices S *on carriers* $\alpha \in SIN_{n-2}$ *reduced to some saturated cardinal* χ *which has the property*

$$\chi^* \leq \chi \wedge Lj^{<\alpha}(\chi). \qquad \dashv$$

It should be noted that there exist cardinals of this kind, for instance, the cardinal $\chi = \chi^*$ for every $\alpha > \chi^*$, $\alpha \in SIN_{n-2}$, as one can see it from lemmas 5.5, 3.8 (for $n-1$ as n).
This agreement makes it possible to apply lemma 4.6 about the spectrum type to these cardinals.
Let us *fix this cardinal* χ up to a special remark, so the lower index χ will be dropped as a rule; for example, the prejump cardinal $\alpha_\chi^{\Downarrow}$ will be denoted simply by $\alpha^{\Downarrow}$ and so on.

Next lemma analyzes the set of matrix disseminators with bases exceeding this matrix order type itself.

Lemma 6.10

Let a matrix S on a carrier α be reduced to the cardinal χ, and let it possesses some disseminator of the level m with a base $\in [OT(S), \chi^+[$.
Then S possesses all disseminators of this level with all bases $< \chi^+$ on this carrier; analogously for generating and floating disseminators.

Proof. Suppose it is wrong, then there is some

$$\rho_0 \in [OT(S), \chi^+[$$

such that for every $\rho \in [\rho_0, \chi^+[$ the matrix S does not possess any disseminator with the base ρ. In this case standing on $\alpha^{\downarrow}$ one can detect the ordinal $\rho_1 \in [OT(S), \chi^+[$ which can be defined through χ by the formula:

$$\exists \delta < \alpha^{\Downarrow}\, SIN_m^{<\alpha^{\Downarrow}}[<\rho_1](\delta) \wedge \forall \rho \in\,]\rho_1, \chi^+[\, \forall \delta < \alpha^{\Downarrow} \neg SIN_m^{<\alpha^{\Downarrow}}[<\rho](\delta)\,.$$

But then by lemma 4.6 $OT(S) \leq \rho_1 < OT(S)$. ⊣

By the similar argument one can easily prove

Lemma 6.11

Let a matrix S on a carrier α be reduced to the cardinal χ, and let it possesses generating disseminators $\check{\delta}^{\rho_0}$, $\check{\delta}^{\rho_1}$ of the level m with bases ρ_0, ρ_1 respectively on this carrier and

$$\rho_0 < OT(S) \leq \rho_1 < \chi^+.$$

Then $$\check{\delta}^{\rho_0} < \check{\delta}^{\rho_1} \quad \text{or} \quad \check{\delta}^{\rho_0} = \check{\delta}^{\rho_1} = \check{\delta}^{\chi^+}.$$

Let us describe some methods of producing δ-matrices that are used in what follows. One should pay very special attention to the following important condition

$$A_n^{\triangleleft \alpha_1}(\chi^*) = \left\| u_n^{\triangleleft \alpha_1}(\underline{l}) \right\| \ ,$$

which plays essential role everywhere further and means that no Σ_n-proposition $\varphi(\underline{l})$ has jump ordinals after χ^* below α_1.

Lemma 6.12 (About singular matrices producing)
Let cardinals $\chi < \alpha_0 < \alpha_1$ *fulfill conditions:*

(i) $\alpha_0 \in SIN_{n-1}^{<\alpha_1}$;

(ii) $A_n^{\triangleleft \alpha_1}(\chi^*) = \| u_n^{\triangleleft \alpha_1}(\underline{l}) \|$.

Then there exists the matrix S_0 *which is reduced to the cardinal* χ *and singular on the carrier* $\alpha'_0 \in]\alpha_0, \alpha_1[$ *such that* $\alpha_0 = \alpha'^{\Downarrow}_{0\chi}$. *In this case the matrix* S_0 *is named produced by the cardinal* α_0 *below* α_1.

Proof. Let us apply lemma 5.12. By this lemma the Π_n-proposition φ

$$\forall \chi' \forall \alpha'_0 \big((\chi' \text{ is a limit cardinal } > \omega_0) \longrightarrow \exists \alpha'_1 > \alpha'_0 \ \sigma(\chi', \alpha'_1) \big).$$

holds; hence it holds below every Π_{n-1}-cardinal due to lemma 3.2 (for $n-1$ as n); but, moreover, it holds also below any cardinal α_1 which satisfies conditions of this lemma.
Justification of this fact is provided by condition (ii), which implies the preservation of this proposition φ below α_1. It is not hard to see, that in the opposite case the Σ_n-statement $\neg\varphi$ receives the jump ordinal *after* χ^* *below* α_1 – namely, the *minimal* pair

(χ', α_0') such that the statement

$$\big((\chi' \text{ is a limit cardinal } > \omega_0) \ \wedge\ \forall \alpha_1' > \alpha_0' \ \neg\sigma(\chi', \alpha_1')\big)$$

holds below α_1; therefore the Σ_n-universal statement $u_n(\underline{l})$ receives this jump ordinal as well due to lemma 2.7. It implies the following violation of condition (ii):

$$A_n^{\triangleleft\alpha_1}(\chi^*) < \|u_n^{\triangleleft\alpha_1}(\underline{l})\| \ .$$

So, $\varphi^{<\alpha_1}$ holds; from here it comes the existence of the cardinal

$$\alpha_0' = \min\{\alpha \in \,]\alpha_0, \alpha_1[\ : \exists S \ \sigma(\chi, \alpha, S)\}$$

and of the matrix S_0 on the carrier α_0'; this minimality provides $\alpha_0 = \alpha_{0\chi}'^{\Downarrow}$. [7] ⊣

Lemma 6.13 (About δ-matrices producing).
Let cardinals $\chi < \alpha_0 < \alpha_1$ *fulfill conditions:*

(i) $\alpha_0 \in SIN_{n-1}^{<\alpha_1}$ *;*

(ii) $A_n^{\triangleleft\alpha_1}(\chi^*) = \|u_n^{\triangleleft\alpha_1}(\underline{l})\|$ *;*

(iii) $\alpha_0 = \sup(\alpha_0 \cap \sup SIN_{m-1}^{<\alpha_1})$ *;*

(iv) on α_0 *some formula* $\psi^{\triangleleft\alpha_0}(\beta, \gamma)$
defines a function $f(\beta) = \gamma < \chi^+$ *which is nondecreasing up to* χ^+.

Then there exists the matrix S_0 *produced by* α_0 *below* α_1*, which is reduced to the cardinal* χ*, singular on the carrier* α_0'

and which possesses all generating disseminators on this carrier:

$$\check{\delta}^{\rho} \in SIN_{m-1}^{<\alpha_0} \quad , \quad \check{\delta}^{\rho} < \alpha_0$$

of the level m *with all bases* $\rho < \chi^+$.
In this case we say that the δ*-matrix* S_0 *is produced by* α_0 *below* α_1.
If in addition $\psi \in \Sigma_m$, *then the set of these disseminators is cofinal to* α_0.

Proof. Recall, that by lemma 6.12 α_0 is the prejump cardinal $\alpha_0'^{\Downarrow}$; due to lemma 6.10 it is enough to establish the existence of the disseminator $\delta^{\rho_0} \in SIN_{m-1}^{<\alpha_0}$ with the base $\rho_0 = OT(S_0)$. From *(iii)*, *(iv)* it comes that there exists a cardinal

$$\beta^S \in SIN_{m-1}^{<\alpha_0}, \;\; \beta^S > \chi, \quad \text{for which} \quad \gamma^S = f(\beta^S) > \rho_0;$$

from lemmas 6.4, 4.6 it follows that the ordinal

$$\beta^S \in SIN_m^{<\alpha_0}[<\rho_0]\,.$$

is the required disseminator.
Really, suppose it is wrong, then there exists some Σ_{m-1}-formula $\varphi(\alpha, \overrightarrow{a})$ having the train of constants $< \rho_0$ such that

$$\forall \alpha < \beta^S \varphi^{\triangleleft \alpha_0}(\alpha, \overrightarrow{a}) \wedge \exists \alpha \in [\beta^S, \alpha_0[\;\; \neg \varphi^{\triangleleft \alpha_0}(\alpha, \overrightarrow{a})\;. \qquad (6.6)$$

Let us take for some brevity $\overrightarrow{a}$ consisting of the one $\alpha_1' < \rho_0$ only. It should be used advantage of the jump cardinal

$$\alpha_2 \in dom\left(\widetilde{\mathbf{S}}_n^{\sin \triangleleft \alpha_0' \overline{\overline{\ }}} \lceil \chi\right),$$

where, recall, α_0' is the minimal carrier $> \alpha_0$ of S_0 and

$$OT\left(\alpha_2 \cap dom\left(\widetilde{\mathbf{S}}_n^{\sin \lhd \alpha'_0}\overline{\overline{\lceil}}\chi\right)\right) = \alpha'_1 \ ; \qquad (6.7)$$

the existence of such α_2 is evident from definition 5.7. After that using (6.6), (6.7) one should define the ordinal $\gamma \in [\gamma^S, \chi^+[$ through χ, α_0, α_2 by the following formula

$$\exists \alpha', \beta < \alpha_0 \ (OT\left(dom\left(\widetilde{\mathbf{S}}_n^{\sin \lhd \alpha_2}\overline{\overline{\lceil}}\chi\right)\right) = \alpha' \wedge \forall \alpha < \beta \ \ \varphi^{\lhd \alpha_0}(\alpha, \alpha') \wedge$$

$$\wedge \neg \varphi^{\lhd \alpha_0}(\beta, \alpha') \wedge \psi^{\lhd \alpha_0}(\beta, \gamma)) \ .$$

But then lemma 4.6 implies the contradiction:

$$OT(S_0) < \gamma < OT(S_0) \ .$$

So, β^S is the disseminator with the base ρ_0 and, hence S_0 possesses generating disseminators with all bases $\rho < \chi^+$.

Now suppose that $\psi \in \Sigma_m$ but, on the contrary, the set of disseminators is not cofinal to α_0. By lemma 6.10, 6.8 3) in this case there exists the disseminator

$$\beta_1 \in SIN_m^{<\alpha_0}[< \chi^+]$$

having the base χ^+. For $\gamma_1 = f(\beta_1)$ the cardinal β_1 extends up to α_0 the Π_m-proposition

$$\forall \beta, \gamma \quad (\psi(\beta, \gamma) \longrightarrow \gamma < \gamma_1)$$

and the function f becomes bounded by $\gamma_1 < \chi^+$ contrary to (iv). $\dashv$

The similar reasoning establishes the following important

Lemma 6.14 (About δ-matrix functions producing)
Let cardinals $\chi < \alpha_1$ *fulfill the conditions:*

(i) $A_n^{\triangleleft\alpha_1}(\chi^*) = \|u_n^{\triangleleft\alpha_1}(\underline{l})\|$;

(ii) $\alpha_1 = \sup SIN_{m-1}^{<\alpha_1} = \sup SIN_{n-1}^{<\alpha_1}$;

(iii) $cf(\alpha_1) > \chi^+$;

(iv) the cardinal $\chi \geq \chi^*$ *is definable through* χ^* *by a formula of the class* $\Sigma_{n-2} \cup \Pi_{n-2}$.

Then there exists the cardinal $\gamma_{\tau_0}^{<\alpha_1}$ *such that on the set*

$$T = \{\tau : \gamma_{\tau_0}^{<\alpha_1} \leq \gamma_\tau^{<\alpha_1}\}$$

there is defined the function $\mathfrak{A}$ *which possesses properties:*

1) *for every* $\tau \in T$ $\mathfrak{A}(\tau)$ *is the* δ*-matrix* S *of the level* m *reduced to* χ *and admissible on some carrier* $\alpha \in \,]\gamma_\tau^{<\alpha_1}, \gamma_{\tau+1}^{<\alpha_1}[$ *for* $\gamma_\tau^{<\alpha_1}$ *below* α_1*;*

2) *the set of all generating disseminators of* S *on* α *of this level is cofinal to* $\alpha_\chi^{\Downarrow}$*;*

3) $cf(\alpha_\chi^{\Downarrow}) = \chi^+$*;*

4) $(\gamma_\tau^{<\alpha_1} + 1) \cap SIN_{m-1}^{<\alpha_1} \subseteq SIN_{m-1}^{<\alpha_\chi^{\Downarrow}}$;

In such case we say that the function $\mathfrak{A}$ *is produced by the cardinal* α_1 *and name it the matrix* δ*-function.*

Proof consists in the repetition of the function $S_f^{<\alpha_1}$ construction from definition 5.14 slightly changed. The upper index $< \alpha_1$ will be omitted for shortness. Let us apply lemma 6.12 to every cardinal $\gamma_{\tau_1} > \chi$, $\gamma_{\tau_1} \in SIN_{m-1}$ of the cofinalily $\neq \chi^+$; the existence of such cardinals follows from (iii) and lemma 3.4 1) (for $n = m-1$). Evidently the matrix S on the carrier $\alpha \in]\gamma_{\tau_1}, \gamma_{\tau_{1+1}}[$ produced by γ_{τ_1} possesses the following property for $\tau = \tau_1$:

$$cf(\alpha^{\Downarrow}) \neq \chi^+ \wedge \forall \gamma \leq \gamma_\tau \left(SIN_{m-1}(\gamma) \longrightarrow SIN_{m-1}^{<\alpha^{\Downarrow}}(\gamma) \right) . \quad (6.8)$$

By lemma 3.2 about restriction (for $n-1$ instead of n) the carriers α of this kind exist in every $[\gamma_\tau, \gamma_{\tau+1}[$ for $\chi < \gamma_{\tau+1} \leq \gamma_{\tau_1}$ (see the proof of lemma 5.17 2) (ii)). Really, let us consider the Π_{n-2}-formula $\varphi(\alpha, \chi, \gamma_\tau^m, \gamma_\tau, S)$:

$$\gamma_\tau < \alpha \wedge cf(\alpha^{\Downarrow}) \neq \chi^+ \wedge \sigma(\chi, \alpha, S) \wedge SIN_{m-1}^{<\alpha^{\Downarrow}}(\gamma_\tau^m)$$

where

$$\gamma_\tau^m = \sup\left((\gamma_\tau + 1) \cap SIN_{m-1}\right).$$

The Σ_{n-1}-proposition $\exists \alpha \ \varphi$ has constants

$$\chi, \quad \gamma_\tau^m, \quad \gamma_\tau < \gamma_{\tau+1}, \quad S \lhd \gamma_{\tau+1},$$

so the SIN_{n-1}-cardinal $\gamma_{\tau+1}$ restricts this proposition and the matrix S appears on the carrier $\alpha \in [\chi_\tau, \chi_{\tau+1}[$ that fulfills (6.8). After that let us repeat definition 5.14 of the function S_f below α_1 but imposing the additional requirement (6.8) upon the carrier α; as a result we obtain the function $S_f^m = (S_\tau^m)_\tau$ taking values:

$$S_\tau^m = \min_{\trianglelefteq}\{S : \exists \alpha < \alpha_1 (\gamma_\tau < \alpha \wedge cf(\alpha^{\Downarrow}) \neq \chi^+ \wedge \sigma(\chi, \alpha, S) \wedge$$

$$\wedge \forall \gamma \leq \gamma_\tau \ (SIN_{m-1}(\gamma) \longrightarrow SIN_{m-1}^{<\alpha^{\Downarrow}}(\gamma)))\} .$$

This function $\preceq$-nondecreases just as the function S_f in lemma 5.17, so due to *(iii)* there exists an ordinal $\tau^m \in dom\left(S^m_f\right)$ such that

$$\gamma_{\tau^m} \in SIN_{m-1} \quad \text{and for every} \quad \tau \geq \tau^m \quad S^m_\tau = S^m_{\tau^m} \ .$$

Let us consider an arbitrary $\tau > \tau^m$ and the carrier $\alpha^m_\tau \in [\gamma_\tau, \gamma_{\tau+1}[$ of the matrix S^m_τ. The definition of S^m_f should be repeated below $\alpha^1_\tau = \alpha^{m\Downarrow}_\tau$ and the resulting function $S^{m<\alpha^1_\tau}_f$ is still $\preceq$-nondecreases and coincides with S^m_f on τ. The reasoning from the end of the previous section shows with the help of condition (iv), that it $\preceq$-nondecreases up to χ^+ and, so, appears the ordinal

$$\alpha_{0\tau} = sup\left\{\gamma^{<\alpha^1_\tau}_{\tau'} : \tau' \in dom\left(S^{m<\alpha^1_\tau}_f\right)\right\}$$

such that

$$\alpha_{0\tau} = sup\left(\alpha_{0\tau} \cap SIN^{<\alpha^1_\tau}_{m-1}\right).$$

It is obvious that $cf(\alpha_{0\tau}) = \chi^+$, hence condition $cf(\alpha^1_\tau) \neq \chi^+$ implies

$$\gamma_\tau < \alpha_{0\tau} < \alpha^1_\tau.$$

Now we can apply lemma 6.13 to $\alpha_{0\tau}$, α^1_τ as α_0, α_1 respectively and to the function

$$f(\beta) = sup\left\{Od\left(S^{m<\alpha^1_\tau}_{\tau'}\right) : \gamma^{<\alpha^1_\tau}_{\tau'} < \beta\right\} \ .$$

which meets all requirements of this lemma.

The matrix $S = S_{0\tau}$ produced by $\alpha_{0\tau}$ on the carrier $\alpha'_{0\tau}$ possesses generating disseminators of the level m with arbitrary bases $< \chi^+$. It is not hard to see that the set of these disseminators is cofinal to $\alpha_{0\tau} = \alpha'^{\Downarrow}_{0\tau}$ and that it has some generating disseminator $\check{\delta} < \gamma_\tau$ admissible for γ_τ.

It comes out that the function $\mathfrak{A}(\tau) = S_{0\tau}$ fulfills all conclusions of this lemma completely. $\dashv$

Now the basic theory is developed sufficiently for the further analysis of matrix functions, which will be undertaken in Part II of this work forthcoming. [8)]

Comments

1) p. 22. Here the light should be shed upon the following points.

First, one can ask, whether in this investigation inaccessible cardinal hypothesis is used in the essential way.
This hypothesis is used through all the text. The Power Set Axiom is used in L_k at almost every step; in particular, when working with Skolem functions while formulas and their spectra are transformed, and in every case providing some new notion.
Besides, the inaccessibility of the cardinal k is used strongly in the k-chain property of algebra B (lemma 1.1), without which Boolean values, spectra and matrices are classes in L_k, not sets, and they can not be compared with each other in a required way. As a result basic theory and special theory both become impossible in this situation.

One can ask also, whether the entire argument can be carried out strictly in ZFC, with inaccessible k replaced by the proper class ordinal k in ZFC (strictly, of course, meaning this proper class in Neumann-Gödel-Bernays Set theory). One can imagine that there is no problem with introducing the Lévy forcing in this context, and, so, that it provides the inconsistency of ZF.
However, the direct transference of the whole reasoning from L_k onto L is considerably hampered. The point is that several important parts of the proof are conducted by using generic extensions of $\mathfrak{M}$ produced by $\mathfrak{M}$-generic ultrafilters (or functions) over (ω_0, k)-algebra Levy B. This algebra has the properties significant for the reasoning, the k-chain property and others, provided by the inaccessibility of k; in there turn they provide the basic properties of spectra, subinnaccessibles, reduced spectra, matri-

ces, matrix functions, etc. It is important that in this extensions every infinite ordinal $< k$ is countable due to first component ω_0. The cardinal ω_0 can be replaced here by some cardinal $\lambda < k$ (k-chain property will remain, etc.). But assume that Levy forcing as a proper class forcing can be used in *this context*, that is k is replaced by the class of all ordinals, – then it provides the generic extension of L in which *every infinite cardinal is still countable*. It is possible to get along without these extensions and work within Boolean-valued universe L^B but still the analogous obstacle should arise – there is the value

$$\|\text{every infinite ordinal is countable}\| = 1$$

(or $\|\text{every ordinal has power} \leq \lambda\| = 1$ if we use λ instead of ω_0) in B. The author intends to clarify this aspect more later.

[2] p. 30. The two-dimensional spectrum can be constructed in the following way. Let us consider the proposition

$$\varphi(\overrightarrow{a}, \underline{l}) = \exists x \ \varphi_1(x, \overrightarrow{a}, \underline{l}) = \exists x \forall y \ \varphi_2(x, y, \overrightarrow{a}, \underline{l}).$$

To each jump ordinal α of φ on the point $\overrightarrow{a}$ below α_1 we adjoin the spectrum of the proposition $\varphi_1' = \exists y \neg \varphi_2(F^{\underline{l}}(\alpha), y, \overrightarrow{a}, \underline{l})$ but on the point $(\alpha, \overrightarrow{a})$. As a result we obtain the function that is two-dimensional spectrum:

$$\mathbf{S}^{\lhd \alpha_1}_{\varphi,2}(\overrightarrow{a}) = \{(\alpha, \beta, \Delta^{\lhd \alpha_1}_{\varphi}(\alpha, \overrightarrow{a}), \Delta^{\lhd \alpha_1}_{\varphi_1'}(\alpha, \beta, \overrightarrow{a})) : \alpha, \beta < \alpha_1 \wedge \\ \wedge \Delta^{\lhd \alpha_1}_{\varphi}(\alpha, \overrightarrow{a}) > 0 \wedge \Delta^{\lhd \alpha_1}_{\varphi_1'}(\alpha, \beta, \overrightarrow{a}) > 0\}.$$

Its first and second projection constitutes the two-dimensional ordinal spectrum, while third and fourth projection constitutes the two-dimensional Boolean spectrum of φ on the point $\overrightarrow{a}$ below

α_1. The three-dimensional spectrum are introduced in a similar way by consequent adjoining the spectra of φ_2 and so on. However the investigation of multi-dimensional spectra lies outside the limits of this work.

[3)] p. 39. Obviously, the subinaccessibility notion can be reformulated in terms of elementary equivalence but in the following substantially more complicated and artificial form:

α is subinaccessible of level n below α_1 *iff* $L_\alpha[l|\alpha_1]$ is Σ_n-elementary substructure of $L_{\alpha_1}[l|\alpha_1]$ *for every* $\mathfrak{M}$-*generic function* l.
Precisely this is done in lemma 3.2. Here few words must be said in connection with this aspect.
This notion can not be reduced naturally to the elementary equivalence of constructive segments L_α, L_{α_1} only, but requires involving all their generic extensions of this kind.
So, using these terms, when parameters α, α_1 are varying simultaneously along with many other cardinals that are subinaccessible also, one receives some multilayer and cumbersome description of the notion which is especially unnatural, since in fact this notion works within L only.
Moreover, the notion of disseminator (§6), which is simple generalization of the subinaccessibility notion, becomes unnaturally complicated in terms of elementary substructures.

Therefore some more convenient description is required, pointing directly to the very essence of the phenomenon, simply definable and therefore more suitable for the investigation of the problem – that is subinaccessibility introduced above.

[4)] p. 48. For this purpose all Boolean values of multi-dimensional spectra should be reduced to certain cardinal χ. For example two-dimensional spectrum (see comment 2)) transforms to its the following reduced form:

$$\mathbf{S}^{\triangleleft\alpha_1}_{\varphi,2}(\overrightarrow{a})\overline{\overline{\lceil}}\chi = \{(\alpha,\beta,\Delta^{\triangleleft\alpha_1}_{\varphi}(\alpha,\overrightarrow{a})\overline{\lceil}\chi, \Delta^{\triangleleft\alpha_1}_{\varphi'_1}(\alpha,\beta,\overrightarrow{a})\overline{\lceil}\chi):$$

$$: \alpha,\beta < \alpha_1 \wedge \ \Delta^{\triangleleft\alpha_1}_{\varphi}(\alpha,\overrightarrow{a})\overline{\lceil}\chi > 0 \ \wedge \ \Delta^{\triangleleft\alpha_1}_{\varphi'_1}(\alpha,\beta,\overrightarrow{a})\overline{\lceil}\chi > 0\}.$$

[5] p. 54. For instance, it is possible to require the definability of $\overline{\delta}$ not in L_k, but in $L_k[l]$; it is possible to weaken condition 2) up to the condition

$$SIN_{n-2}(\overline{\alpha}_1) \wedge OT(\,]\overline{\alpha}_0,\overline{\alpha}_1[\, \cap SIN^{<\overline{\alpha}_1}_{n-1}) \geq \overline{\chi}^+,$$

also the condition imposed on $\overline{\chi}$ can be weakened substantially, etc.

[6] p. 87. Here one should point out the following interesting notion of base nonexcessiveness: *the admissible base* ρ *is named nonexcessive on the carrier* α, iff the decreasing of ρ implies the decreasing of generating disseminator for S on the same carrier α:

$$\forall\rho'\left(\rho' < \rho \longrightarrow \check{\delta}^{\rho'} < \check{\delta}^{\rho}\right).$$

The point is that among bases of the matrix S disseminators on the carrier α there are possible bases ρ which possess certain "excessive" information in the sense that for some $\rho' < \rho$ still

$$\check{\delta}^{\rho'} = \check{\delta}^{\rho}$$

on α and, so, instead of ρ it *can be used the smaller base* $\rho' < \rho$ *without any loss* for the location of the generating disseminator (and, hence, for its action). It is natural to use bases free of such excessiveness, that are nonexcessive bases. One should note here that the using of some smaller base, may be, is possible, *but only* when the corresponding disseminator decreases.

It is not hard to prove the lemma which establishes the remarkable property: not only such base defines the location of generating disseminator, but, inversely, such disseminator defines its nonexcessive base even for different matrices on different carriers:

Lemma

Let $S_1, \ S_2$ *be matrices on carriers* $\alpha_1, \ \alpha_2$ *which possess generating disseminators* $\check{\delta}_1, \ \check{\delta}_2$ *of the level* m *with nonexcessive bases* $\rho_1, \ \rho_2$ *on these carriers respectively, then:*

$$if \quad \check{\delta}_1 = \check{\delta}_2 \in SIN_{m-1}^{<\alpha_1^{\Downarrow}} \cap SIN_{m-1}^{<\alpha_2^{\Downarrow}} \quad then \quad \rho_1 = \rho_2.$$

[7] p. 90. Here it is quite necessary to pay attention to the fact, that condition (ii) of this lemma

$$A_n^{\triangleleft\alpha_1}(\chi^*) = \left\| u_n^{\triangleleft\alpha_1}(\underline{l}) \right\|$$

justifies the above-stated proof strongly, and all other forthcoming reasoning as well. To illustrate in what can result misunderstanding of this condition the author would like to present (for an example) the report of one referee in 2000:

Referee: "Lemma 6.12 is false. Here is a counterexample. Let χ be the ordinal χ^* defined on page 25 [here definition 5.4 on page 59 — author]. Then for any α_1, condition (ii) of 6.12 holds. Let α_1 be the *least* ordinal α satisfying the following three properties:

(a) $\alpha > \chi$;
(b) $\alpha \in SIN_{n-2}$;
(c) $\sup \left\{ \beta < \alpha : \beta \in SIN_{n-1}^{<\alpha} \right\} = \alpha$.

Clearly such an α exists, and any such α satisfies condition (i) of 6.12. Let α_0 be any member of $SIN_{n-1}^{<\alpha_1}$ greater than χ.

Thus the hypothesis of 6.12 is satisfied. But the conclusion of 6.12 is violated. For if S_0 and α'_0 satisfy the conclusion, then by definition of "singular matrix" and "carrier", α'_0 satisfies (a)-(c); but $\alpha'_0 < \alpha_1$ contradicting the minimality of α_1.

The mistake in the author's proof of 6.12 involves the use of Lemma 5.12. Now 5.12 is, indeed, true in the model L_k, but it is not true in the model L_{α_1}. To rephrase this in the author's notation, if ϕ is the statement of 5.12, then ϕ is true but $\phi^{<\alpha_1}$ is not true. ϕ is — as the author points out in the proof of 5.12 — a Π_n proposition, and α_1 is merely a level $n-2$, not a level $n-1$, subinaccessible; so there is no justification for claiming that ϕ can be restricted to α_1. The counterexample shows that it cannot be." [here ϕ means φ from the proof of lemma 6.12 of this work - author].

Commentary of the author in his answer in 2000: "No, this counterexample is false: not for every α_1 condition (ii) holds; for example, for α_1 used by the referee! Really, the referee is right that in this case the statement ϕ of 5.12, used by him, is true in L_k (and therefore preserves below χ^*) and is not true below α_1 but he makes the wrong conclusion. Vice versa, the opposite situation holds: the statement $\neg\phi$ receives the jump cardinal after χ^* below α_1; therefore the Σ_n-universal statement $u_n(\underline{l})$ receives such cardinal as well (moreover, infinitely many such cardinals in conditions of the counterexample). It means the violation of condition (ii), that is in reality

$$A_n^{\triangleleft\alpha_1}(\chi^*) < \left\| u_n^{\triangleleft\alpha_1}(\underline{l}) \right\| .$$

Yes, indeed, "ϕ is a Π_n proposition, and α_1 is merely a level $n-2$, not a level $n-1$", but still ϕ preserves below α_1 if condition (ii) fulfills."

[8] p. 96. This section can be finished by one more comment on the disposition of disseminators. In lemmas 6.13, 6.14 δ-matrices produced on the carriers α have generating disseminators that are disposed cofinally to $\alpha^{\Downarrow}_{\chi}$ and $cf(\alpha^{\Downarrow}_{\chi}) = \chi^{+}$. Using methods of reasoning from proofs of lemmas 6.13, 6.14 one can prove that for $\underline{\prec}$-minimal δ-matrices and $m \geq n+1$ this is unavoidable:

Lemma

Let cardinals $\chi < \gamma_{\tau}^{<\alpha_1}$ *and a matrix* S *fulfill the conditions:*

(i) $A_n^{\lhd\alpha_1}(\chi^*) = \|u_n^{\lhd\alpha_1}(\underline{l})\|$ *;*

(ii) S *is the* δ*-matrix of the level* $m \geq n+1$ *admissible on some carrier* $\alpha \in]\gamma_{\tau}^{<\alpha_1}, \alpha_1[$ *for* $\gamma_{\tau}^{<\alpha_1}$ *below* α_1*;*

(iii) S *is* $\underline{\prec}$*-minimal of all* δ*-matrices with this property.*

Then S *on the carrier* α *possesses generating disseminators of the level* m *with all bases* $\rho < \chi^{+}$ *disposed cofinally to* $\alpha^{\Downarrow}_{\chi}$ *and* $cf(\alpha^{\Downarrow}_{\chi}) = \chi^{+}$.

This lemma is not used in this work and so the proof is omitted.

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